Developments in Mathematics

Vladimir Gutlyanskii
Vladimir Ryazanov
Uri Srebro
Eduard Yakubov

The Beltrami Equation

A Geometric Approach

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Uri Srebro • Eduard Yakubov

The Beltrami Equation

A Geometric Approach

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Vladimir Gutlyanskii
Department of the Partial
Differential Equations
Institute of Applied Mathematics
and Mechanics
National Academy of Sciences of Ukraine
Donetsk, Ukraine

Vladimir Ryazanov
Department of the Function Theory
Institute of Applied Mathematics
and Mechanics
National Academy of Sciences of Ukraine
Donetsk, Ukraine

Uri Srebro
Faculty of Mathematics
Technion-Israel Institute of Technology
Haifa, Israel

Eduard Yakubov
Faculty of Sciences
H.I.T. – Holon Institute of Technology
Holon, Israel

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Dedicated to Professor Bogdan Bojarski
who is an important contributor to the
Beltrami equation theory

Preface

The book is a summation of many years' work on the study of general Beltrami equations with singularities. This is not only a summary of our own long-term collaboration but also with that of many other authors in the field. We show that our geometric approach based on the modulus and capacity developed by us makes it possible to derive the main known existence theorems, including sophisticated and more general existence theorems that have been recently established.

The Beltrami equation plays a significant role in geometry, analysis, and physics, and, in particular, in the theory of quasiconformal mappings and their generalizations, Kleinian groups, and Teichmüller spaces. There has been renewed interest and activity in these areas and, in particular, in the study of degenerate and alternating Beltrami equations since the early 1990s.

In this monograph, we restrict ourselves to the study of very basic properties of solutions in the degenerate and in the alternating cases like existence, uniqueness, distortion, boundary behavior, and mapping problems that can be derived by extremal length methods. The monograph can serve as a textbook for a one- or two-semester graduate course.

Donetsk, Ukraine — Vladimir Gutlyanskii
Donetsk, Ukraine — Vladimir Ryazanov
Haifa, Israel — Uri Srebro
Holon, Israel — Eduard Yakubov

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Chapter 1
Introduction

1.1 The Beltrami Equation

Let $\mathbb{C}$ be the complex plane. In the complex notation $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$, the *Beltrami equation* in a domain $D \subset \mathbb{C}$ has the form

$$w_{\bar{z}} = \mu(z) w_z, \tag{B}$$

where $\mu : D \to \mathbb{C}$ is a measurable function and

$$w_{\bar{z}} = \overline{\partial} w = \frac{1}{2}(w_x + \mathrm{i}w_y) \quad \text{and} \quad w_z = \partial w = \frac{1}{2}(w_x - \mathrm{i}w_y)$$

are formal derivatives of w in $\bar{z}$ and z, while w_x and w_y are partial derivatives of w in the variables x and y, respectively. For the geometric interpretation of μ, see Appendix A.4.5.

In the real variables x, y, u, and v, (B) can be written in the form of the system

$$\begin{cases} v_y & = \alpha u_x + \beta u_y \\ -v_x & = \beta u_x + \gamma u_y, \end{cases} \tag{B$'$}$$

where α, β, and γ are given measurable functions in x and y; see, e.g., [263]. For $\mu \equiv 0$, (B$'$) reduces to the Cauchy–Riemann system, i.e., (B$'$) with $\beta \equiv 0$ and $\alpha = \gamma \equiv 1$.

This book is devoted mainly to the Beltrami (B). In addition to the theory of the Beltrami (B), there is a theory of the *Beltrami equation of the second kind*

$$w_{\bar{z}} = \nu(z) \cdot \overline{w_z}, \tag{S}$$

with applications to many problems of mathematical physics; see, for instance, [136]. The Beltrami equation of the second type also plays a significant role in the theory of harmonic mappings in the plane; see, e.g., [59, 201]. Hence, we give also

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some results on the Beltrami equation with two characteristics:

$$w_{\bar{z}} \;=\; \mu(z)\cdot w_z \;+\; \nu(z)\cdot \overline{w_z}\,. \tag{T}$$

The existence problem for degenerate Beltrami (B) when

$$K_\mu(z) \;:\;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;\notin\; L^\infty$$

is currently an active area of research; see, e.g., [26, 48–50, 56, 57, 65, 66, 70, 76, 77, 99, 117, 140, 147, 159, 160, 174, 212–214, 238, 254, 271]. The study of such homeomorphisms started from the theory of the so-called mean quasiconformal mappings; see, e.g., [5, 41, 94, 95, 100, 133, 134, 138, 139, 141, 185, 187, 204, 206, 243, 244, 257, 273, 274], and related to the modern theory of mappings with finite distortion; see, e.g., [25, 42, 104, 105, 112–114, 116, 118, 125–130, 132, 155, 161–165, 181–183, 192–195, 207–209, 225].

1.2 Historical Remarks

The system (B′) first appeared in [86] in connection with finding isothermal coordinates on a surface. Local coordinates u and v on a given surface are called *isothermal* if the curves $u=\text{const}$ are orthogonal to the curves $v=\text{const}$ or, equivalently, if the length element $\mathrm{d}s$ is given by

$$\mathrm{d}s^2 = \Lambda(u,v)\;(\mathrm{d}u^2+\mathrm{d}v^2).$$

The transition from given local coordinates x and y to isothermal coordinates u and v is an injective mapping $(x,y)\to(u,v)$ satisfying

$$a(x,y)\,\mathrm{d}x^2+2b(x,y)\,\mathrm{d}x\mathrm{d}y+c(x,y)\,\mathrm{d}y^2=\Lambda(\mathrm{d}u^2+\mathrm{d}v^2),\quad \Lambda>0,$$

where u and v are solutions of the Beltrami system (B′) with

$$\alpha=\frac{b}{\sqrt{\Delta}},\;\beta=\frac{a}{\sqrt{\Delta}},\gamma=\frac{c}{\sqrt{\Delta}}\qquad\text{and}\qquad \Delta=ac-b^2\geq\Delta_0>0.$$

Gauss in [86] proved the existence and uniqueness of a solution in the case of real analytic α,β, and γ, or, equivalently, when μ is real-analytic. The equation then appeared in the Beltrami studies on surface theory; see [32]. There is a long list of names associated with proofs of existence and uniqueness theorems for more general classes of μ's. Among them we mention A. Korn [124] and L. Lichtenstein [154] (Hölder continuous μ's, 1916) and M.A. Lavrent'ev [144] (continuous μ's, 1935). C.B. Morrey [176] was the first to prove the existence and uniqueness of a homeomorphic solution for a measurable μ (1939). Morrey's proof was based on PDE methods. In the mid-1950s, L. Ahlfors [6], B. Bojarski [43, 44], and I.N. Vekua [263] proved the existence in the measurable case by singular integral methods. L. Ahlfors and L. Bers [7] established the analytic dependence of solutions on parameters; cf. the paper [44] and its English translation in [47], and the corresponding discussion in the monograph [51].

1.3 Applications of Beltrami Equations

The Beltrami equation was first used in various areas such as differential geometry on surfaces (see Sect. 1.2), hydrodynamics, and elasticity. Most applications of the Beltrami equation are based on the close relation to quasiconformal (qc) mappings. Plane qc mappings appeared already implicitly in the late 1920s in papers by Grötzsch [97]. The relation between qc mappings and the Beltrami equation was noticed by Ahlfors [4] and Lavrent'ev [144] in the 1930s. The significant connection between the theory of the Beltrami equation and the theory of plane qc mappings has stimulated intensive study and enriched both theories. Most notable is the contribution of qc theory to the modern development of Teichmüller spaces and Kleinian groups.

The Beltrami equation turned out to be useful in the study of Riemann surfaces, Teichmüller spaces, Kleinian groups, meromorphic functions, low dimensional topology, holomorphic motion, complex dynamics, Clifford analysis, control theory, and robotics. The following list is only a partial list of references: [2,4–6,8–11, 22,23,26,27,33–38,45,60,72,78,80–82,90,117,135–137,143,146,151,157,232, 242,245,246,263]. Part of the list consists of books and expository papers where further references can be found. For the classical theory of the Beltrami equation and plane qc mappings, we refer to [9,30,44,152].

1.4 Classification of Beltrami Equations

We say that μ is *bounded* in D if $||\mu||_\infty < 1$ and that μ is *locally bounded* in D if $\mu|A$ is bounded whenever A is a relatively compact subdomain of D. The study of the Beltrami equation is divided into three cases according to the nature of $\mu(z)$ in D:

(1) *The classical case*: $||\mu||_\infty < 1$.
(2) *The degenerate case*: $|\mu| < 1$ almost everywhere (a.e.) and $||\mu||_\infty = 1$.
(3) *The alternating case*: $|\mu| < 1$ a.e. in a part of D and $1/|\mu| < 1$ a.e. in the remaining part of D.

1.5 ACL Solutions

By writing $f : D \to \mathbb{C}$, we assume that D is a domain in $\mathbb{C}$, which is an open and connected set and that f is continuous. A mapping $f : D \to \mathbb{C}$ is *absolutely continuous on lines (ACL)*, $f \in$ ACL, if for every rectangle $R, \overline{R} \subset D$, whose sides are parallel to the coordinate axes, f is absolutely continuous on almost every horizontal and almost every vertical line; see, e.g., [9] or [152]. A function $f : D \to \mathbb{C}$ is a *solution of (B)*, if f is ACL in D, and its ordinary partial derivatives, which exist a.e. in D, satisfy (B) a.e. in D.

Some authors (cf. [99,116]) include in the definition of a solution the assumption that f belongs to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and that (B) holds in the sense of distributions. If $f \in W^{1,1}_{\mathrm{loc}}$ and f is continuous, then $f \in$ ACL, and the generalized (distributional) partial derivatives coincide with the ordinary partial derivatives. In general, an ACL function need not belong to $W^{1,1}_{\mathrm{loc}}$. For some μ's, however, every ACL solution is just a $W^{1,1}_{\mathrm{loc}}$ solution.

A solution $f : D \to \mathbb{C}$ of (B) which is a homeomorphism of D into $\mathbb{C}$ is called a *μ-homeomorphism* or *μ-conformal* mapping. In the above cases (1) and (2), a solution $f : D \to \mathbb{C}$ of (B) will be called *elementary* if f is open and discrete, meaning that f maps every open set onto an open set and that the preimage of every point in D consists of isolated points. Elementary solutions of (B) are also called by us *μ-regular* mappings.

If $f : D \to \mathbb{C}$ is open and has partial derivatives a.e. in D, then by a result of Gehring and Lehto (see [9, 91, 152]), see also with the earlier Menchoff result for homeomorphisms in [172,255,256], f is differentiable a.e. in D. It thus follows that every elementary solution is differentiable a.e.

Let $f : D \to \mathbb{C}$ be an elementary solution. The *complex dilatation* of f is defined by

$$\mu_f(z) = \mu(z) = \bar{\partial} f(z)/\partial f(z), \tag{1.5.1}$$

if $\partial f(z) \neq 0$ and by $\mu(z) = 0$ if $\partial f(z) = 0$. For such a mapping, the *dilatation* is

$$K_f(z) := K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}. \tag{1.5.2}$$

Note that $K_f < \infty$ a.e. if and only if $|\mu(z)| < 1$ a.e., and that $K_f \in L^\infty$ if and only if $||\mu||_\infty < 1$.

1.6 Ellipticity of the Beltrami Equation

In the classical case (1) in Sect. 1.4, the system (B) is *uniformly* or, in a different terminology, *strongly elliptic*, i.e.,

$$\Delta = \alpha\gamma - \beta^2 \geq \Delta_0 > 0, \tag{1.6.1}$$

and in the relaxed classical case (2) in Sect. 1.5, (B) is *elliptic*, i.e.,

$$\Delta = \alpha\gamma - \beta^2 \geq 0. \tag{1.6.2}$$

Chapter 2
Preliminaries

2.1 BMO Functions in $\mathbb{C}$

The class BMO was introduced by John and Nirenberg in the paper [122] and soon became an important concept in harmonic analysis, partial differential equations, and related areas; see, e.g., [21, 24, 84, 103, 200] and [239].

A real-valued function u in a domain D in $\mathbb{C}$ is said to be of *bounded mean oscillation* in D, $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$, and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| \, \mathrm{d}m(z) < \infty, \tag{2.1.1}$$

where the supremum is taken over all discs B in D, $\mathrm{d}m(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$, $|B|$ is the Lebesgue measure of B, and

$$u_B = \frac{1}{|B|} \int_B u(z) \, \mathrm{d}m(z).$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

A function φ in BMO is said to have *vanishing mean oscillation* (abbreviated as $\varphi \in VMO$), if the supremum in (2.1.1) taken over all disks B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO was introduced by Sarason in [227]. A large number of papers are devoted to the existence, uniqueness and properties of solutions for various kinds of differential equations and, in particular, of elliptic type with coefficients of the class VMO; see, e.g., [61, 119, 166, 184, 190].

If $u \in \mathrm{BMO}$ and c is a constant, then $u + c \in \mathrm{BMO}$ and $\|u\|_* = \|u + c\|_*$. Obviously $L^\infty \subset \mathrm{BMO}$.

John and Nirenberg [122] established the following fundamental fact (see also [103]):

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Lemma 2.1. *If u is a nonconstant function in* BMO(D)*, then*

$$|\{z\in B: |u(z)-u_B|>t\}| \le a\mathrm{e}^{-\frac{b}{\|u\|_*}\cdot t}\cdot|B| \tag{2.1.2}$$

for every disc B in D and all $t>0$, where a and b are absolute positive constants which do not depend on B and u. Conversely, if $u\in L^1_{\mathrm{loc}}$ and if for every disc B in D and for all $t>0$

$$|\{z\in B: |u(z)-u_B|>t\}| < a\mathrm{e}^{-bt}|B| \tag{2.1.3}$$

for some positive constants a and b, then $u\in$ BMO(D).

We will need the following lemma which follows from Lemma 2.1:

Lemma 2.2. *If u is a nonconstant function in* BMO(D)*, then*

$$|\{z\in B: |u(z)|>\tau\}| \le A\mathrm{e}^{-\beta\tau}\cdot|B| \tag{2.1.4}$$

for every disc B in D and all $\tau>|u_B|$, where

$$\beta = b/\|u\|_* \quad \textit{and} \quad A = a\mathrm{e}^{b|u_B|/\|u\|_*}, \tag{2.1.5}$$

and the constants a and b are as in Lemma 2.1.

Proof. For $t>0$, let $\tau=t+|u_B|$, $D_1=\{z\in B: |u(z)|>\tau\}$, and $D_2=\{z\in B: |u(z)-u_B|>t\}$. Then, by the triangle inequality, $D_1\subset D_2$, and hence, by (2.1.2),

$$|D_1|\le|D_2|\le a\mathrm{e}^{b|u_B|/\|u\|_*}\cdot\mathrm{e}^{-\tau b/\|u\|_*}\cdot|B|,$$

which implies (2.1.4) with A and β as in (2.1.5). □

Proposition 2.1. BMO $\subset L^p_{\mathrm{loc}}$ *for all* $p\in[1,\infty)$.

Proof. Let $u\in$ BMO. Then by (2.1.4),

$$\int\limits_B |u(z)|^p\mathrm{d}m(z) \le |B|\{|u_B|^p + A\int\limits_{|u_B|}^{\infty} t^p\mathrm{e}^{-\beta t}\mathrm{d}t\} < \infty. \qquad \square$$

Remark 2.1. Given a domain D, $D\subset\mathbb{C}$, there is a nonnegative real-valued function u in D such that $u(z)\le Q(z)$ a.e. for some $Q(z)$ in BMO(D) and $u\notin$ BMO(D). For $D=\mathbb{C}$, one can take, for instance, $Q(x,y)=1+|\log|y||$ and $u(x,y)=Q(x,y)$ if $y>0$ and $u(x,y)=1$ if $y\le 0$.

2.2 BMO Functions in $\overline{\mathbb{C}}$

Later on, we will need also several facts about BMO functions on $\overline{\mathbb{C}}$ and their relations to BMO functions on $\mathbb{C}$.

We identify $\overline{\mathbb{C}}$ with the unit sphere

$$S^2 = \{x \in R^3 : |x| = 1\}$$

and the functions on $\overline{\mathbb{C}}$ with the functions on S^2. This is done with the aid of the stereographic projection P of S^2 onto $\overline{\mathbb{C}}$ which is given for $(x_1, x_2, x_3) \in S^2 \setminus (0,0,1)$ by

$$z = P(x_1, x_2, x_3) = \frac{x_1 + \mathrm{i}x_2}{1 - x_3}.$$

A real-valued measurable function u in a domain $D \subset \overline{\mathbb{C}}$ is said to be in $\mathrm{BMO}(D)$, if u is locally integrable with respect to the spherical area and

$$\|u\|_{*\sigma} = \sup_B \frac{1}{\sigma(B)} \int_B |u - u_B| \, \mathrm{d}\sigma < \infty, \tag{2.2.1}$$

where the supremum is taken over all spherical discs B in D. Here, $\sigma(B)$ denotes the spherical area of B, $\mathrm{d}\sigma = 4\mathrm{d}x_1\mathrm{d}x_2/(1 + x_1^2 + x_2^2)^2$, and

$$u_B = \frac{1}{\sigma(B)} \int_B u \, \mathrm{d}\sigma. \tag{2.2.2}$$

The following two lemmas enable one to decide whether a function u in a domain $D \subset \overline{\mathbb{C}}$ belongs to $\mathrm{BMO}(D)$ (in the spherical sense) by considering the restriction u_0 of u to $D_0 = D \setminus \{\infty\}$ (see [200], p. 7):

Lemma 2.3. $u \in \mathrm{BMO}(D)$ *if* $u_0 \in \mathrm{BMO}(D_0)$. *Furthermore,*

$$c^{-1}\|u_0\|_* \le \|u\|_{*\sigma} \le c\|u_0\|_*, \tag{2.2.3}$$

where c is an absolute constant.

The following lemma is a consequence of Lemmas 2.2 and 2.3:

Lemma 2.4. *If either* $u \in \mathrm{BMO}(D)$ *or* $u_0 \in \mathrm{BMO}(D_0)$, *then for* $\tau > \gamma$,

$$\sigma\{z \in B : |u(z)| > \tau\} \le \alpha \mathrm{e}^{-\beta\tau} \tag{2.2.4}$$

for every spherical disc B in D where the constants α, β, *and* γ *depend on B as well as on the function u.*

Proof. If $\overline{B} \in \mathbb{C}$, then by Lemma 2.3, we have (2.1.4), and since $\sigma(E) \le 4|E|$ for every measurable set $E \subset \mathbb{C}$, (2.2.4) follows. If $\infty \in \overline{B}$, then for suitable rotation R of S^2, ∞ is exterior to $\overline{B'}$, $B' = R(B)$, and the assertion follows by Lemma 2.3 and the validity of (2.2.4) with B' and $\hat{u} = u \circ R^{-1}$ instead of B and u. Now, in view of the invariance of the spherical area with respect to rotations, by Lemmas 2.2 and 2.3, we have again (2.2.4). □

2.2.1 Removability of Isolated Singularities of BMO Functions

The following lemma holds for the BMO functions and cannot be extended to the $\mathrm{BMO}_{\mathrm{loc}}$ functions (see [200], p. 5):

Lemma 2.5. *Let E be a discrete set in a domain D, $D \subset \mathbb{C}$, and let u be a function in $\mathrm{BMO}(D \setminus E)$. Then any extension $\hat{u}$ of u on D is in $\mathrm{BMO}(D)$ and $\|u\|_* = \|\hat{u}\|_*$.*

2.2.2 BMO Functions, qc Mappings, qc Arc, and Symmetric Extensions

We say that a Jordan curve E in $\overline{\mathbb{C}}$ is a *K-quasicircle* if $E = f(\partial\Delta)$ for some K-quasiconformal map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$; see Sect. 3.1 below. A curve E is a *qc* curve if it is a subarc of a quasicircle. The following lemma is a special case of a theorem by Reimann on the characterization of quasiconformal maps in R^n, $n \geq 2$, in terms of the induced isomorphism on BMO (see [199], p. 266):

Lemma 2.6. *If f is a K-qc map of a domain D in $\mathbb{C}$ onto a domain D' and $u \in \mathrm{BMO}(D')$, then $v = u \circ f$ belongs to $\mathrm{BMO}(D)$, and*

$$\|v\|_* \leq c\|u\|_*$$

where c is a constant which depends only on K.

The next lemma can be found in [121].

Lemma 2.7. *Let D be a Jordan curve such that ∂D is a K-quasicircle, and let u be a $\mathrm{BMO}(D)$ function. Then u has an extension $\hat{u}$ on $\mathbb{C}$ which belongs to $\mathrm{BMO}(\mathbb{C})$ and*

$$\|\hat{u}\|_* \leq c\|u\|_*,$$

where c depends only on K.

The following lemmas, which concern symmetric extensions of BMO functions, will be needed in studying the reflection principle and boundary behavior of the so-called BMO-quasiconformal and BMO-quasiregular mappings in Chap. 5. It should be noted that these lemmas cannot be extended to $\mathrm{BMO}_{\mathrm{loc}}$ functions (see [200], p. 8).

Lemma 2.8. *If $u \in \mathrm{BMO}(\mathbb{D})$ and $\hat{u}$ is an extension of u on $\mathbb{C}$, which satisfies the symmetry condition*

$$\hat{u}(t) = \begin{cases} u(z) & \text{if } z \in \mathbb{D}, \\ u(z/|z|^2) & \text{if } z \in \mathbb{C}\setminus\mathbb{D}, \end{cases}$$

then $\hat{u} \in \mathrm{BMO}(\mathbb{C})$ and $\|u\|_* = \|\hat{u}\|_*$.

Similarly, one can prove the following lemma:

Lemma 2.9. *Let D be a domain in $\mathbb{C}$, E a free boundary arc in ∂D which is either a line segment or a circular arc, D^* a domain which is symmetric to D with respect to the corresponding line or circle such that $D \cap D^* = \emptyset$ and $\Omega = D \cup D^* \cup E$. If $u \in \mathrm{BMO}(D)$ and $\hat{u}$ is an extension of u on Ω which satisfies the symmetry condition*

$$\hat{u}(z) = \begin{cases} u(z) & \text{if } z \in D, \\ u(z^*) & \text{if } z \in D^*, \end{cases}$$

then $\hat{u} \in \mathrm{BMO}(\Omega)$ and $\|\hat{u}\|_* = \|u\|_*$.

2.3 FMO Functions

Let D be a domain in the complex plane $\mathbb{C}$. Following [112] and [113], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon \to 0}} \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \; \infty, \tag{2.3.1}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z) \, \mathrm{d}m(z) \tag{2.3.2}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0,\varepsilon)$. Condition (2.3.1) includes the assumption that φ is integrable in some neighborhood of the point z_0. We call $d_\varphi(z_0)$ the *dispersion* of the function φ at the point z_0. We say also that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation in D*, abbreviated as $\varphi \in$ FMO(D) or simply $\varphi \in FMO$, if φ has a finite dispersion at every point $z_* \in D$.

Remark 2.2. Note that if a function $\varphi : D \to \mathbb{R}$ is integrable over $B(z_0,\varepsilon_0) \subset D$, then

$$\fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; \leq \; 2 \cdot \overline{\varphi}_\varepsilon(z_0), \tag{2.3.3}$$

and the left-hand side in (2.3.3) is continuous in the parameter $\varepsilon \in (0,\varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0,\varepsilon_0)$,

$$\sup_{\varepsilon \in [\delta_0,\varepsilon_0]} \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \infty. \tag{2.3.4}$$

If (2.3.1) holds, then

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \infty. \tag{2.3.5}$$

The value of the left-hand side of (2.3.5) is called the *maximal dispersion* of the function φ in the disk $B(z_0, \varepsilon_0)$.

Proposition 2.2. *If, for some collection of numbers* $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon \to 0}} \;\; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \; \mathrm{d}m(z) \; < \infty, \tag{2.3.6}$$

then φ *is of finite mean oscillation at* z_0*.*

Proof. Indeed, by the triangle inequality,

$$⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \; \mathrm{d}m(z) \; \leq \; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \; \mathrm{d}m(z) \; + \; |\varphi_\varepsilon - \overline{\varphi}_\varepsilon(z_0)|$$

$$\leq \; 2 \cdot ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \; \mathrm{d}m(z) \,. \qquad \square$$

Choosing in Proposition 2.2, in particular, $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we obtain the following:

Corollary 2.1. *If, for a point* $z_0 \in D$,

$$\overline{\lim_{\varepsilon \to 0}} \; ⨍_{B(z_0,\varepsilon)} |\varphi(z)| \; \mathrm{d}m(z) \; < \; \infty, \tag{2.3.7}$$

then φ *has finite mean oscillation at* z_0*.*

Recall that a point $z_0 \in D$ is called a *Lebesgue point* of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \;\; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \; \mathrm{d}m(z) \; = \; 0. \tag{2.3.8}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point. We thus have the following corollary:

Corollary 2.2. *Every function* $\varphi : D \to \mathbb{R}$*, which is locally integrable, has a finite mean oscillation at almost every point in* D*.*

Remark 2.3. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ (see, e.g., [191], p. 5) and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (2.3.7) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and $\mathrm{BMO}(D) \neq \mathrm{BMO}_{\mathrm{loc}}(D)$. Also, $\mathrm{BMO}_{\mathrm{loc}}(D) \neq \mathrm{FMO}(D)$ as is clear from the following examples.

2.3.1 Examples of Functions $\varphi \in FMO \setminus BMO_{loc}$

Set $z_n = 2^{-n},\ r_n = 2^{-pn^2},\ p > 1,\ c_n = 2^{2n^2}$, $D_n = \{z \in \mathbb{C} : |z - z_n| < r_n\}$, and

$$\varphi(z) = \sum_{n=1}^{\infty} c_n \chi_{D_n}(z),$$

where χ_E denotes the *characteristic function* of a set E, i.e.,

$$\chi_E(z) = \begin{cases} 1, & z \in E, \\ 0, & \text{otherwise.} \end{cases}$$

It is evident by Corollary 2.1 that $\varphi \in \mathrm{FMO}(\mathbb{C} \setminus \{0\})$.

To prove that $\varphi \in \mathrm{FMO}(0)$, fix N such that $(p-1)N > 1$, and set $\varepsilon = \varepsilon_N = z_N + r_N$. Then $\bigcup_{n \geq N} D_n \subset \mathbb{D}(\varepsilon) := \{z \in \mathbb{C} : |z| < \varepsilon\}$ and

$$\begin{aligned} \int_{\mathbb{D}(\varepsilon)} \varphi &= \sum_{n \geq N} \int_{D_n} \varphi = \pi \sum_{n \geq N} c_n r_n^2 \\ &= \sum_{n \geq N} 2^{2(1-p)n^2} < \sum_{n \geq N} 2^{2(1-p)n} \\ &< C \cdot [2^{(1-p)N}]^2 < 2C\varepsilon^2. \end{aligned}$$

Hence, $\varphi \in \mathrm{FMO}(0)$ and, consequently, $\varphi \in \mathrm{FMO}(\mathbb{C})$.

On the other hand,

$$\int_{\mathbb{D}(\varepsilon)} \varphi^p = \pi \sum_{n > N} c_n^p \cdot r_n^2 = \sum_{n > N} 1 = \infty.$$

Hence, $\varphi \notin L^p(\mathbb{D}(\varepsilon))$, and therefore, $\varphi \notin \mathrm{BMO_{loc}}$ by Proposition 2.1.

We conclude this section by constructing functions $\varphi : \mathbb{C} \to \mathbb{R}$ of the class $C^\infty(\mathbb{C} \setminus \{0\})$ which belongs to FMO but not to L^p_{loc} for any $p > 1$ and hence not to $\mathrm{BMO_{loc}}$. In the following example, $p = 1 + \delta$ with an arbitrarily small $\delta > 0$. Set

$$\varphi_0(z) = \begin{cases} e^{\frac{1}{|z|^2 - 1}}, & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1. \end{cases} \tag{2.3.9}$$

Then φ_0 belongs to C_0^∞ and hence to $\mathrm{BMO_{loc}}$. Consider the function

$$\varphi_\delta^*(z) = \begin{cases} \varphi_k(z), & \text{if } z \in B_k, \\ 0, & \text{if } z \in \mathbb{C} \setminus \bigcup B_k, \end{cases} \tag{2.3.10}$$

where $B_k = B(z_k, r_k),\ z_k = 2^{-k},\ r_k = 2^{-(1+\delta)k^2}$, $\delta > 0$, and

$$\varphi_k(z) = 2^{2k^2} \varphi_0 \left(\frac{z - z_k}{r_k} \right), \quad z \in B_k,\ k = 2, 3, \dots . \tag{2.3.11}$$

Then φ^*_δ is smooth in $\mathbb{C} \setminus \{0\}$ and thus belongs to $\mathrm{BMO_{loc}}(\mathbb{C} \setminus \{0\})$ and hence to $\mathrm{FMO}(\mathbb{C} \setminus \{0\})$.

Now,

$$\int\limits_{B_k} \varphi_k(z)\, \mathrm{d}m(z) \ = \ 2^{-2\delta k^2} \int\limits_{\mathbb{C}} \varphi_0(z)\, \mathrm{d}m(z). \tag{2.3.12}$$

Hence,

$$\overline{\lim_{\varepsilon \to 0}} \ ⨍_{B(\varepsilon)} \varphi^*_\delta(z)\, \mathrm{d}m(z) \ < \ \infty. \tag{2.3.13}$$

Thus, $\varphi \in \mathrm{FMO}$ by Corollary 2.1.

Indeed, by setting

$$K = K(\varepsilon) = \left[\log_2 \frac{1}{\varepsilon} \right] \le \log_2 \frac{1}{\varepsilon}, \tag{2.3.14}$$

where $[A]$ denotes the integral part of the number A, we have

$$J = ⨍_{D(\varepsilon)} \varphi^*_\delta(z)\, \mathrm{d}m(z) \le I \cdot \sum_{k=K}^{\infty} 2^{-2\delta k^2} / \pi 2^{-2(K+1)}, \tag{2.3.15}$$

where $I = \int_{\mathbb{C}} \varphi(z)\, \mathrm{d}m(z)$. If $K\delta > 1$, i.e., $K > 1/\delta$, then

$$\sum_{k=K}^{\infty} 2^{-2\delta k^2} \le \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K} \sum_{k=0}^{\infty} \left(\frac{1}{4} \right)^k = \frac{4}{3} \cdot 2^{-2K}, \tag{2.3.16}$$

i.e., $J \le 16I/3\pi$.

On the other hand,

$$\int\limits_{B_k} \varphi_k^{1+\delta}(z)\, \mathrm{d}m(z) = \int\limits_{\mathbb{C}} \varphi_0^{1+\delta}(z)\, \mathrm{d}m(z) \tag{2.3.17}$$

and hence $\varphi^*_\delta \notin L^{1+\delta}(U)$ for any neighborhood U of 0.

2.4 On Sobolev's Classes

We recall the necessary definitions and basic facts on L^p, $p \in [1,\infty]$, and the Sobolev spaces $W^{l,p}$ $l = 1,2,\ldots$. Given an open set U in $\mathbb{R}^n$ and a positive integer l, $C_0^l(U)$ denotes a collection of all functions $\varphi : U \to \mathbb{R}$ with compact support having all partial continuous derivatives of order at least l in U; $\varphi \in C_0^\infty(U)$ if $\varphi \in C_0^l(U)$ for all $l = 1,2,\ldots$. A vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ with natural coordinates is called a *multi-index*. Every multi-index α is associated with the differential operator $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Now, let u and $v : U \to \mathbb{R}$ be locally integrable functions. The function v is called the *distributional derivative* $D^\alpha u$ of u if

$$\int_U u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_U v \varphi \, \mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(U), \tag{2.4.1}$$

where the notation dx corresponds to the Lebesgue measure in $\mathbb{R}^n$.

The concept of the generalized derivative was introduced by Sobolev in [231]. The *Sobolev class* $W^{l,p}(U)$ consists of all functions $u : U \to \mathbb{R}$ in $L^p(U)$, $p \geq 1$, with generalized derivatives of order l summable of order p. A function $u : U \to \mathbb{R}$ belongs to $W^{l,p}_{\mathrm{loc}}(U)$ if $u \in W^{l,p}(U_*)$ for every open set U_* with compact closure $\overline{U_*} \subset U$. A similar notion is introduced for vector functions $f : U \to \mathbb{R}^m$ in the componentwise sense.

A function $\omega : \mathbb{R}^n \to \mathbb{R}$ with a compact support in the unit ball $\mathbb{B}^n$ is called a *Sobolev averaging kernel* if ω is nonnegative and belongs to $C_0^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \omega(x) \, \mathrm{d}x \; = \; 1. \tag{2.4.2}$$

The well-known example of such a function is $\omega(x) = \gamma\varphi(|x|^2 - \frac{1}{4})$, where $\varphi(t) = \mathrm{e}^{1/t}$ for $t < 0$ and $\varphi(t) \equiv 0$ for $t \geq 0$, and the constant γ is chosen so that (2.4.2) holds. Later on, we use only ω depending on $|x|$.

Let U be a nonempty bounded open subset of $\mathbb{R}^n$ and $f \in L^1(U)$. Extending f by zero outside of U, we set

$$f_h = \omega_h * f = \int_{|y|\leq 1} f(x+hy)\,\omega(y)\,\mathrm{d}y = \frac{1}{h^n}\int_U f(z)\,\omega\left(\frac{z-x}{h}\right)\mathrm{d}z, \tag{2.4.3}$$

where $f_h = \omega_h * f$, $\omega_h(y) = \omega\,(y/h)\,,\, h > 0$, is called the *Sobolev mean functions* for f. It is known that $f_h \in C_0^\infty(\mathbb{R}^n)$, $\|f_h\|_p \leq \|f\|_p$ for every $f \in L^p(U)$, $p \in [1,\infty]$, and $f_h \to f$ in $L^p(U)$ for every $f \in L^p(U)$, $p \in [1,\infty)$; see, e.g., 1.2.1 in [170]. It is clear that if f has a compact support in U, then f_h also has a compact support in U for small enough h.

A sequence $\varphi_k \in L^1(U)$ is called *weakly fundamental* if

$$\lim_{k_1,k_2\to\infty}\int\limits_U \Phi(x)\left(\varphi_{k_1}(x)-\varphi_{k_2}(x)\right)\,\mathrm{d}x = 0 \qquad \forall\ \Phi\in L^\infty(U). \tag{2.4.4}$$

It is well known that the space $L^1(U)$ is *weakly complete*, i.e., every weakly fundamental sequence $\varphi_k \in L^1(U)$ *weakly converges* in $L^1(U)$, i.e., there is a function in $\varphi \in L^1(U)$ such that

$$\lim_{k\to\infty}\int\limits_U \Phi(x)\ \varphi_k(x)\ \mathrm{d}x = \int\limits_U \Phi(x)\ \varphi(x)\ \mathrm{d}x \qquad \forall\ \Phi\in L^\infty(U); \tag{2.4.5}$$

see, e.g., Theorem IV.8.6 in [73]:

Recall also the following statement (see, e.g., Theorem 1.2.5 in [110]).

Proposition 2.3. *Let f and $g \in L^1_{\mathrm{loc}}(U)$. If*

$$\int f\,\varphi\ \mathrm{d}x = \int g\,\varphi\ \mathrm{d}x \qquad \forall\ \varphi\in C_0^\infty(U), \tag{2.4.6}$$

then $f = g$ a.e.

The following fact is known for the Sobolev classes $W^{1,p}(U)$, $p>1$; see, e.g., Lemma III.3.5 in [202], also Theorem 4.6.1 in [79]. Note that this fact for $p=2$ was known long ago in the plane for the so-called mappings with the bounded Dirichlet integral; see, e.g., Theorem 1 in [247].

Lemma 2.10. *Let U be a bounded open set in $\mathbb{R}^n$ and let $f_k : U \to \mathbb{R}$ be a sequence of functions of the class $W^{1,p}(U)$, $p>1$. Suppose that the norm sequence $\|f_k\|_{1,p}$ is bounded and $f_k \to f$ as $k\to\infty$ in $L^1(U)$. Then $f\in W^{1,p}(U)$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$ as $k\to\infty$ weakly in $L^p(U)$.*

Here we apply instead of Lemma 2.10, which is not valid for $p=1$, the following lemma; see, e.g., [222] or [220].

Lemma 2.11. *Let U be a bounded open set in $\mathbb{R}^n$ and let $f_k : U\to\mathbb{R}$ be a sequence of functions of the class $W^{1,1}(U)$. Suppose that $f_k \to f$ as $k\to\infty$ weakly in $L^1(U)$, $\partial f_k/\partial x_j$, $k=1,2,\ldots$, $j=1,2,\ldots,n$ are uniformly bounded in $L^1(U)$ and their indefinite integrals are absolutely equicontinuous. Then $f\in W^{1,1}(U)$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$ as $k\to\infty$ weakly in $L^1(U)$.*

The proof of Lemma 2.11 is based on Proposition 2.3 and the above mentioned criterion of weak convergence in the space L^1.

Remark 2.4. The weak convergence $f_k \to f$ in $L^1(U)$ implies that

$$\sup_k\ \|f_k\|_1\ <\ \infty;$$

see, e.g., IV.8.7 in [73]. The latter together with

$$\sup_k \|\partial f_k/\partial x_j\|_1 < \infty,$$

$j = 1,2,\ldots,n$, implies that $f_k \to f$ by the norm in L^q for every $1 \le q < n/(n-1)$, the limit function f belongs to $BV(U)$, the class of functions of bounded variation, but, generally speaking, not to the class $W^{1,1}(U)$; see, e.g., Remark in 4.6 and Theorem 5.2.1 in [79]. Thus, the additional condition of Lemma 2.11 on absolute equicontinuity of the indefinite integrals of $\partial f_k/\partial x_j$ is essential; cf. also Remark to Theorem I.2.4 in [197].

Proof of Lemma 2.11. It is known that the space L^1 is weakly complete; see Theorem IV.8.6 in [73]. Thus, it suffices to prove that the sequences $\partial f_k/\partial x_j$ are weakly fundamental in $L^1(U)$.

Indeed, by the definition of generalized derivatives, we have that

$$\int_U \varphi(x)\, \frac{\partial f_k}{\partial x_j}\, \mathrm{d}x = -\int_U f_k(x)\, \frac{\partial \varphi}{\partial x_j}\, \mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(U). \tag{2.4.7}$$

Note that the integrals on the right-hand side in (2.4.7) are bounded linear functionals in $L^1(U)$ and the sequence f_k is weakly fundamental in $L^1(U)$ because $f_k \to f$ weakly in $L^1(U)$. Hence, in particular,

$$\int_U \varphi(x)\, \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) \mathrm{d}x \to 0 \quad \forall\, \varphi \in C_0^\infty(U)$$

as k_1 and $k_2 \to \infty$.

Now, let $\Phi \in L^\infty(U)$. Then $\|\Phi_h\|_\infty \le \|\Phi\|_\infty$ and $\Phi_h \to \Phi$ in the norm of $L^1(U)$ for its Sobolev mean functions Φ_h, and hence, $\Phi_h \to \Phi$ in measure as $h \to 0$. Set $\varphi_m = \Phi_{h_m}$, where $\Phi_{h_m} \to \Phi$ a.e. as $m \to \infty$. Considering restrictions of Φ to compacta in U, we may assume that $\varphi_m \in C_0^\infty(U)$. By the Egoroff theorem, $\varphi_m \to \Phi$ uniformly on a set $S \subset U$ such that $|U \setminus S| < \delta$, where $\delta > 0$ can be arbitrary small; see, e.g., III.6.12 in [73]. Given $\varepsilon > 0$, we have that

$$\left| \int_S (\Phi(x) - \varphi_m(x)) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) \mathrm{d}x \right|$$
$$\le 2 \cdot \max_{x \in S} |\, \Phi(x) - \varphi_m(x)\, | \cdot \sup_{k=1,2,\ldots} \int_U \left| \frac{\partial f_k}{\partial x_j} \right| \mathrm{d}x \le \frac{\varepsilon}{3}$$

for all large enough m. Choosing one such m, we have that

$$\left| \int_U \varphi_m(x)\, \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right) \mathrm{d}x \right| \le \frac{\varepsilon}{3}$$

for k_1 and k_2 large enough. By absolute equicontinuity of the indefinite integrals of $\partial f_k/\partial x_j$, there is $\delta > 0$ such that

$$\int_E \left| \frac{\partial f_k}{\partial x_j} \right| \mathrm{d}x \le \frac{1}{12} \frac{\varepsilon}{\|\Phi\|_\infty}$$

for all $k = 1, 2, \ldots$ and every measurable set $E \subset U$ with $|E| < \delta$; see IV.8.10 and IV.8.11 in [73]. Setting $E = U \setminus S$, we obtain that

$$\left| \int_U \Phi(x) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) \mathrm{d}x \right| \le I_1 + I_2 + I_3,$$

where

$$I_1 = \left| \int_E (\Phi(x) - \varphi_m(x)) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) \mathrm{d}x \right|,$$

$$I_2 = \left| \int_S (\Phi(x) - \varphi_m(x)) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) \mathrm{d}x \right|,$$

$$I_3 = \left| \int_U \varphi_m(x) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) \mathrm{d}x \right|,$$

and hence, by the above arguments,

$$\left| \int_U \Phi(x) \left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j} \right) \mathrm{d}x \right| \le \varepsilon$$

for large enough k_1 and k_2. Thus, $(\partial f_k)/(\partial x_j)$ is weakly fundamental in $L^1(U)$ and hence $(\partial f_k)/(\partial x_j)$ converges weakly in $L^1(U)$ just to $(\partial f)/(\partial x_j)$ by (2.4.7); see Proposition 2.3. □

Recall that a function $f : D \to \mathbb{C}$ is absolutely continuous on lines, abbreviated as $f \in$ ACL, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL (possibly modified on a set of Lebesgue measure zero) if it belongs to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ of locally integrable functions with locally integrable first generalized derivatives and, conversely, if $f \in$ ACL has locally integrable first partial derivatives, then $f \in W^{1,1}_{\mathrm{loc}}$; see, e.g., 1.2.4 in [170].

A continuous one-to-one mapping f between domains D and D' in $\mathbb{C}$ is called a *homeomorphism* if its inverse mapping f^{-1} is also continuous. Recall also that a mapping $f : D \to \mathbb{C}$ is *open* if $f(A)$ is open whenever A is open, and f is *discrete* if the preimage of every point consists of isolated points in D.

Let $f : D \to \mathbb{C}$ be an ACL open mapping. Since f is ACL, it has partial derivatives f in x and y and the formal derivatives

$$\partial f = f_z = \frac{1}{2}(f_x - \mathrm{i} f_y) \quad \text{and} \quad \overline{\partial} f = f_{\bar{z}} = \frac{1}{2}(f_x + \mathrm{i} f_y),$$

a.e. in D, and since, in addition, f is open, it is differentiable a.e. in D by the Gehring–Lehto result in [91]; see also the corresponding Menchoff result for homeomorphisms in [172], Theorem III.3.1 in [152] and Lemma II.B.1 in [9]. At points z in D where f is differentiable, the *dilatation* $\mu(z) = \mu_f(z)$ is defined by

$$\mu(z) = \overline{\partial} f(z) / \partial f(z),$$

if $\partial f(z) \neq 0$ and by $\mu(z) = 0$ if $\partial f(z) = 0$. If, in addition, f is *sense preserving* (*s.p.*) i.e., its *Jacobian* $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \geq 0$ a.e., then $|\mu(z)| \leq 1$ a.e. and the *dilatation* of f is

$$K_f(z) = K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \geq 1 \quad \text{a.e.} \tag{2.4.8}$$

An ACL embedding with a given μ a.e. is called a μ-*homeomorphism*. In some places, as in [70] and [254], the term is more restricted. Note that a μ-homeomorphism f is *quasiconformal* (*qc*), if $||\mu||_\infty < 1$, or equivalently if $K_\mu \in L^\infty(D)$.

Given a measurable function $Q : D \to [1, \infty)$ and an ACL s.p. open discrete mapping $f : D \to \mathbb{C}$, we say that f is Q(z)-*quasiregular* ($Q(z)$-*qr*) if $K_f(z) \leq Q(z)$ a.e. in D. For completeness, constant mappings are considered as $Q(z)$-*qr* mappings. If, in addition, f is injective, we say that f is Q(z)-*quasiconformal* ($Q(z)$-*qc*). An ACL s.p. open discrete mapping $f : D \to \mathbb{C}$ is BMO-*quasiregular* (BMO-qr) if it is $Q(z)$-*qr* for some Q in $\mathrm{BMO}(D)$, i.e.,

$$K_f(z) \leq Q(z) \text{ a.e.,} \quad Q \in \mathrm{BMO}(D), \tag{2.4.9}$$

with $K_f(z)$ as in (2.4.8). If, in addition, f is injective, it is BMO-*quasiconformal* (BMO-qc).

More generally, if $f : D \to \mathbb{C}$ is $Q(z)$-*qc*, and Q belongs to a given class $\mathscr{F}$ of functions, we say that f is $\mathscr{F}$-*quasiconformal* ($\mathscr{F}$-*qc*), and $\mathscr{F}$-*qr* maps are defined similarly. For example, a mapping $f : D \to \mathbb{C}$ is called FMO-*quasiconformal* (FMO-qc), if f is $Q(z)$-*qc* with some Q in $\mathrm{FMO}(D)$.

Proposition 2.4. *Every homeomorphic* ACL *solution f of the Beltrami equation* (B) *with $K_\mu \in L^1_{\mathrm{loc}}$ belongs to $W^{1,1}_{\mathrm{loc}}$. If $K_\mu \in L^p_{\mathrm{loc}}$, $p \in [1, \infty]$, then $f \in W^{1,s}_{\mathrm{loc}}$ where $s = 2/(1 + 1/p) \in [1, 2]$. In particular, $f \in W^{1,2}_{\mathrm{loc}}$ if $K_\mu \in L^\infty$.*

Proof. Indeed, if f is a homeomorphic ACL solution f of the Beltrami equation (B), then f has partial derivatives f_x and f_y a.e., and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ is nonnegative a.e. and, moreover,

$$|\,\overline{\partial} f| \;\le\; |\,\partial f| \;\le\; |\,\partial f| \,+\, |\,\overline{\partial} f| \;\le\; K_\mu^{1/2}(z) \cdot J_f^{1/2}(z) \qquad \text{a.e.} \tag{2.4.10}$$

Recall that if a homeomorphism $f : D \to \mathbb{C}$ has finite partial derivatives a.e., then

$$\int\limits_B J_f(z)\,\mathrm{d}m(z) \;\le\; |f(B)| \tag{2.4.11}$$

for every Borel set $B \subseteq D$, where $|A|$ denotes the *Lebesgue measure* of a set A in $\mathbb{C}$; see, e.g., Lemma III.3.3 in [152]. Consequently, applying successively the Hölder inequality and the inequality (2.4.11) to (2.4.10), we get

$$\|\partial f\|_s \;\le\; \|K_\mu\|_p^{1/2} \cdot |f(C)|^{1/2}, \tag{2.4.12}$$

where $\|\cdot\|_s$ and $\|\cdot\|_p$ denote the L^s- and L^p-norms in a compact set $C \subset D$, respectively. □

Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have the *Lusin (N) property* if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the *(N^{-1}) property* if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

Proposition 2.5. *If $f \in$ ACL is a homeomorphism of a domain $D \subset \mathbb{C}$ into $\mathbb{C}$ with $f^{-1} \in W^{1,2}_{\mathrm{loc}}$, then f^{-1} is locally absolutely continuous, preserves null sets, and f is regular a.e., i.e., it is differentiable with $J_f(z) \ne 0$ a.e.*

Proof. Indeed, the assertion on f^{-1} follows from the inclusion $f^{-1} \in \mathrm{W}^{1,2}_{\mathrm{loc}}$; see, e.g., Theorem III.4.1 in [152]. As an ACL function, f has a.e. partial derivatives and hence it has a total differential a.e. by the Menchoff theorem in [172]; see also Theorem III.3.1 in [152] and Lemma II.B.1 in [9], cf. [91] for open mappings. Then, by the Ponomarev result in [189], N^{-1} property of f is equivalent to the condition $J_f(z) \ne 0$ for a.e. $z \in D$.

Let us give a more direct proof of the last statement. Let E denote the set of points of D where f is differentiable and $J_f(z) = 0$, and suppose that $|E| > 0$. Then $|f(E)| > 0$, since $E = f^{-1}(f(E))$ and since f^{-1} preserves null sets. Clearly, f^{-1} is not differentiable at any point of $f(E)$, contradicting the fact that f^{-1} is differentiable a.e. □

2.5 Modulus and Capacity

A path γ in $\mathbb{C}$ is a continuous mapping $\gamma : \Delta \to \mathbb{C}$, where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$.Given a family of paths Γ in $\mathbb{C}$, a Borel function $\rho : \mathbb{C} \to [0,\infty]$ is called *admissible function* for Γ, abbreviated as $\rho \in \operatorname{adm}\Gamma$, if

$$\int_\gamma \rho(z)\ |\mathrm{d}z| \geq 1$$

for each $\gamma \in \Gamma$. The (conformal) *modulus* $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \rho^2(z)\ \mathrm{d}m(z)$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \emptyset$, where $\mathrm{d}m(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$. Thus, every family Γ which contains a constant path is of infinite modulus.

We say that a property P holds for *almost every (a.e.)* path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero. We also say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every path in Γ_2 has a subpath which belongs to Γ_1. It is known that $M(\Gamma_1) \geq M(\Gamma_2)$; see, e.g., [85], pp. 176–178, or Theorem 6.4 in [258].

The modulus is a conformal invariant and plays an important role in the study of conformal, qc, and qr mappings; see, e.g., [9, 120, 152, 203, 258] and [267]. We show that the modulus method is very useful in the study of Beltrami equations.

Given a domain D in the extended plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and nonempty disjoint compact sets C_0 and C_1 in D, the triple (C_0, C_1, D) is called a *condenser* in D, and its *capacity* is a number

$$\operatorname{cap}\ (C_0, C_1, D) = \inf \int_D |\nabla u|^2\ \mathrm{d}m(z), \tag{2.5.1}$$

where the infimum is taken over all *ACL* functions $u : D \to \mathbb{R}$ with values 0 and 1 on C_0 and C_1, respectively. We set below $(C_0, C_1) = (C_0, C_1, \bar{\mathbb{C}})$.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Delta(E,F,D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E,\ \gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Delta(E,F) = \Delta(E,F,\overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. It is well known that

$$\operatorname{cap}\ (C_0, C_1, D)\ =\ M(\Delta(C_0, C_1, D)) \tag{2.5.2}$$

for every condenser in $D \subset \mathbb{R}^n$, $n \geq 2$; see, e.g., [88] and [107]. Moreover,

$$M(\Delta(C_0, C_1, R))\ =\ M(\Delta(C_0, C_1)) \tag{2.5.3}$$

for ring domains R and its complementary components C_0 and C_1 in $\overline{\mathbb{C}}$; see, e.g., Theorem 11.3 in [258]. Recall that a *ring domain*, or briefly a *ring* in $\overline{\mathbb{C}}$, is a domain whose complement $\overline{\mathbb{C}} \setminus R$ consists of two connected sets.

For points $z, \zeta \in \overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$, the *spherical (chord) distance* $s(z,\zeta)$ between z and ζ is given by

$$
\begin{aligned}
s(z,\zeta) &= \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta, \\
s(z,\infty) &= \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty.
\end{aligned} \tag{2.5.4}
$$

Given a set $E \subset \overline{\mathbb{C}}$, $\delta(E)$ denotes the *spherical diameter* of E, i.e.,

$$
\delta(E) = \sup_{z_1,z_2 \in E} s(z_1,z_2). \tag{2.5.5}
$$

The following basic inequality (2.5.6) will be used frequently. Let $f : D \to \overline{\mathbb{C}}$ be a qc mapping which is $Q(z)$-*qc* for a given function Q in $L^1_{\rm loc}$, and let Γ be a path family in D and $\rho \in \operatorname{adm} \Gamma$. Then (see V(6.6) in [152] or [153])

$$
M(f\Gamma) \leq \iint_D Q(z)\rho^2(z)\, \mathrm{d}m(z). \tag{2.5.6}
$$

Below, $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Lemma 2.12. *Let $f : \mathbb{D} \to \overline{\mathbb{C}} \setminus \{a,b\}$, $a,b \in \overline{\mathbb{C}}$, $s(a,b) \geq \delta > 0$, be a qc mapping which is $Q(z)$-qc for some Q in $L^1(\mathbb{D})$. Let z_1 be a point in $\mathbb{D} \setminus \{0\}$, and suppose that $s(f(z_1), f(0)) \geq \delta$. Then*

$$
s(f(z), f(0)) \geq \psi(|z|) \tag{2.5.7}
$$

for every point z with $|z| < r = min(1-|z_1|, |z_1|/2)$, where ψ is a positive, strictly increasing function depending only on δ and $||Q||_1$.

Proof. Given z_2 with $|z_2| < r$, choose a continuum E_1 which meets 0 and z_1 and a continuum E_2 which meets the point z_2 and $\partial\mathbb{D}$ such that dist $(E_1, E_2 \cup \partial\mathbb{D}) = |z_2|$; see Fig. 1.

More precisely, denote by L the straight line generated by the pair of points 0 and z_1 and let C be the circle $|z - z_2| = |z_2|$. Denote by t_0 the tangency point to C of the ray starting from z_1 that is opposite to z_2 with respect to L (an arbitrary one of the two possible if $z_2 \in L$). Then $E_1 = [z_1, t_0] \cup \gamma(0, t_0)$, where $\gamma(0, t_0)$ is the shortest arc of C joining 0 and t_0, $E_2 = [z_2, z_*]$, where z_* is the point (opposite t_0 with respect to L) of the intersection of $\partial\mathbb{B}^n$ with the straight line that is perpendicular to L and passing through z_2. Note that z_2 is the nearest point in E_2 for any point z in the half plane that is opposite to z_2 with respect to L. Hence, z_2 is the nearest point in E_2 for E_1.

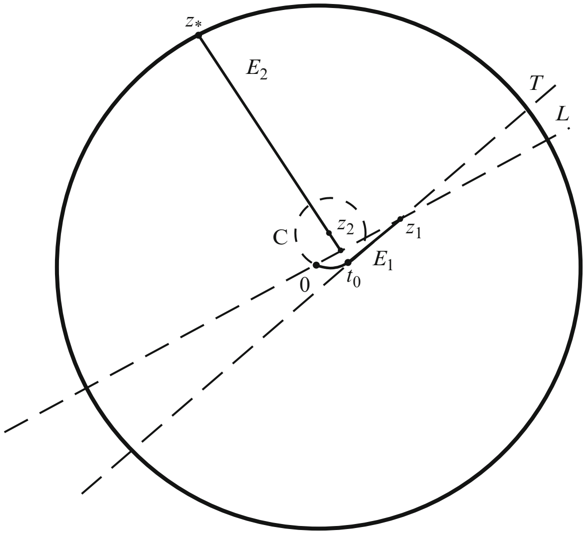


Fig. 1 The ring domain A'

Let Γ denote the family of paths which join E_1 and $E_2 \cup \partial\mathbb{D}$. Then

$$\rho(z) = |z_2|^{-1}\chi_{\mathbb{D}}(z)$$

is admissible for Γ and, by (2.5.6),

$$M(f\Gamma) \leq \iint \rho^2(z)Q(z)\,\mathrm{d}m(z) \leq |z_2|^{-2}\iint_{\mathbb{D}} Q(z)\,\mathrm{d}m(z) = \frac{||Q||_1}{|z_2|^2}. \qquad (2.5.8)$$

The ring domain $A' = f(\mathbb{D}\backslash(E_1 \cup E_2))$, see Fig. 1, separates the continuum $E_1' = f(E_1)$ and $E_2' = \bar{\mathbb{C}}\backslash f(\mathbb{D}\backslash E_2)$, and since

$$s(E_1') \geq s(f(z_1), f(0)) \geq \delta, \;\; s(E_2') \geq s\left(\overline{\mathbb{C}}\backslash f(\mathbb{D})\right) \geq \delta$$

and

$$s(E_1', E_2') \leq s(f(z_2), f(0)),$$

it follows by Theorem 12.7 in [258] that

$$M(f(\Gamma)) \geq \lambda(s(f(z_2), f(0))), \qquad (2.5.9)$$

where $\lambda(t) = \lambda_\delta(t)$ is a strictly decreasing positive function with $\lambda(t) \to \infty$ as $t \to 0$. Hence, by (2.5.8) and (2.5.9), $s(f(z_2), f(0)) \geq \psi(|z|)$ where

$$\psi(t) = \lambda^{-1}\left(\frac{||Q||}{t^2}\right)$$

has the required properties. $\square$

The following lemma will be also useful later on:

Lemma 2.13. *Let D be a domain in $\mathbb{C}$, $z_0 \in D$, f a homeomorphism of $D_0 = D \setminus \{z_0\}$ into $\mathbb{C}$, $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$, $0 < \varepsilon_0 < \operatorname{dist}(z_0, \partial D)$, $C_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$, $\varepsilon \in (0, \varepsilon_0)$, and $A_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$. Then*

$$\delta(E) \leq \frac{32}{\delta(F)} \cdot \exp\left(-\frac{2\pi}{M(\Delta(fC_\varepsilon, fC_0))}\right), \tag{2.5.10}$$

where E is the connected component of $\overline{\mathbb{C}} \setminus fA_\varepsilon$ containing the cluster set

$$C(z_0, f) = \{w \in \overline{\mathbb{C}} : \quad w = \lim_{n \to \infty} f(z_n), \quad z_n \in D_0\}$$

and F is the rest of $\overline{\mathbb{C}} \setminus fA_\varepsilon$.

Proof. Indeed, first of all, we have that

$$\operatorname{cap}(E, F) \geq \operatorname{cap} R_{\mathrm{T}}\left(\frac{1}{\delta(E)\delta(F)}\right), \tag{2.5.11}$$

where $\delta(E)$ and $\delta(F)$ denote the spherical diameters of the continua E and F, respectively, and $R_{\mathrm{T}}(t)$ is the Teichmüller ring

$$R_{\mathrm{T}}(t) = \overline{\mathbb{C}} \setminus ([-1, 0] \cup [t, \infty]), \quad t > 1; \tag{2.5.12}$$

see, e.g., 7.37 in [267]. It is known that

$$\operatorname{cap} R_{\mathrm{T}}(t) = \frac{2\pi}{\log \Phi(t)}, \tag{2.5.13}$$

where the function Φ admits the good estimates

$$t + 1 \leq \Phi(t) \leq 16 \cdot (t + 1) < 32 \cdot t, \quad t > 1; \tag{2.5.14}$$

see, e.g., (7.19) and (7.22) in [267]. Hence, the inequality (2.5.11) implies that

$$\operatorname{cap}(E, F) \geq \frac{2\pi}{\log \frac{32}{\delta(E)\delta(F)}}. \tag{2.5.15}$$

Thus,

$$\delta(E) \leq \frac{32}{\delta(F)} \exp\left(-\frac{2\pi}{\operatorname{cap}(E, F)}\right), \tag{2.5.16}$$

which implies the desired estimate (2.5.10) because the family of paths $\Delta(E, F)$ is minorized by $\Delta(fC_\varepsilon, fC_0)$; see also (2.5.2) and (2.5.3). □

We adopt the following conventions: Given a set $E \in \mathbb{C}$ and a path $\gamma : \Delta \to \mathbb{C}$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If γ is locally rectifiable, we set

$$l(\gamma \cap E) = |E_\gamma|, \tag{2.5.17}$$

where

$$E_\gamma = l_\gamma(\gamma^{-1}(E)). \tag{2.5.18}$$

Here, $|A|$ means the length (Lebesgue) measure of a set $A \subset \mathbb{R}$ and $l_\gamma : \Delta \to \Delta_\gamma$ is as in Sect. 2.1. In general for sets A in $\mathbb{C}$, $|A|$ will denote the Lebesgue measure of A. Note that

$$E_\gamma = \gamma_0^{-1}(E), \tag{2.5.19}$$

where $\gamma_0 : \Delta_\gamma \to \mathbb{C}$ is the natural parametrization of γ and that

$$l(\gamma \cap E) \;=\; \int\limits_{\Delta} \chi_E(\gamma(t))\, \gamma'(t)\, \mathrm{d}t \;=\; \int\limits_{\Delta_\gamma} \chi_{E_\gamma}(s)\, \mathrm{d}s. \tag{2.5.20}$$

We say that $\gamma \cap E$ is measurable on γ if E_γ is measurable in Δ_γ.

Remark 2.5. The definition of the modulus immediately implies that:

1) Almost every path in $\mathbb{C}$ is rectifiable.
2) Given a Borel set B in $\mathbb{C}$ of measure zero,

$$l(\gamma \cap B) = 0 \tag{2.5.21}$$

for a.e. rectifiable path γ in $\mathbb{C}$.
3) For every Lebesgue measurable set E in $\mathbb{R}^n$, there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $|B^* \setminus B_*| = 0$, and thus, by 2), E_γ and χ_{E_γ} is a measurable set, respectively, and a measurable function in the length interval Δ_γ for a.e. γ in $\mathbb{C}$.

The following lemma extends Theorem 33.1 in [258] from Borel sets to arbitrary sets (cf. also Theorem 3 in [85]) and is based on 3):

Lemma 2.14. *Let E be a set in a domain $D \subset \mathbb{C}$. Then E is measurable if and only if $\gamma \cap E$ is measurable for a.e. path γ in D. Moreover, $|E| = 0$ if and only if*

$$l(\gamma \cap E) = 0 \tag{2.5.22}$$

on a.e. path γ in D.

Proof. Suppose first that E is measurable. Then by 3) in Remark 2.5, $\gamma \cap E$ is measurable for a.e. path γ in D.

For the other direction, let C be a closed cube in D with edges parallel to the coordinate axes. By the assumption, $\gamma \cap E$ is measurable for a.e. line segment γ joining opposite faces of C and parallel to the edges. Thus, by the Fubini theorem, E is measurable.

Next, suppose that $|E| = 0$. Then there is a Borel set B such that $|B| = 0$ and $E \subset B$. By Remark 2.5, (2.5.21) and hence (2.5.22) hold for a.e. path γ in D.

The sufficiency of (2.5.22) follows from the corresponding result in Theorem 33.1 in [258] by virtue of 3) in Remark 2.5 This completes the proof. □

Given a Lebesgue measurable function $\rho : \mathbb{C} \to [0,\infty]$, there is a Borel function $\rho^* : \mathbb{C} \to [0,\infty]$ such that $\rho^* = \rho$ a.e. in $\mathbb{R}^n$; see, e.g, 2.3.5 in [83] and [224], p. 69. This suggests an alternative definition of the modulus. A Lebesgue measurable function $\rho : \mathbb{C} \to [0,\infty]$ is *extensively admissible* for a path family Γ in $\mathbb{C}$, abbreviated as $\rho \in \operatorname{ext\,adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\,|dz| \ \geq\ 1 \qquad \text{for a.e. } \gamma \in \Gamma. \tag{2.5.23}$$

Note that (2.5.23) includes the assumption that the function $s \mapsto \rho(\gamma(s))$ is measurable in the interval $[0, l(\gamma)]$; the path is parametrized by arc length. The *extensive modulus* $\overline{M}(\Gamma)$ of Γ is defined as

$$\overline{M}(\Gamma) \ =\ \inf \int\limits_{\mathbb{C}} \rho^2(z)\,dm(z), \tag{2.5.24}$$

where the infimum is taken over all $\rho \in \operatorname{ext\,adm}\Gamma$. It is easy to see that

$$\overline{M}(\Gamma) \ =\ M(\Gamma). \tag{2.5.25}$$

Further, we use the *Hausdorff metric* between compact sets in the plane

$$h(E,F) = \max\left\{ \sup_{z_1\in E}\inf_{z_2\in F}|z_1 - z_2|;\ \sup_{z_2\in F}\inf_{z_1\in E}|z_1 - z_2| \right\}. \tag{2.5.26}$$

Let us give here an elementary proof of the following lemma that will be useful for further applications:

Lemma 2.15. *Let D be a domain in $\mathbb{C}$, E and F be mutually disjoint compact sets in D, and let E_k and F_k, $k = 1,2,\ldots$, be sequences of continua in D such that $E_k \to E$ and $F_k \to F$ as $k \to \infty$ in the Hausdorff metric. Then*

$$M(\Delta(E,F;D)) \ \leq\ \liminf_{k\to\infty} M(\Delta(E_k,F_k;D)). \tag{2.5.27}$$

Proof. If either E or F is degenerated into a point, then $M(\Delta(E,F;D)) = 0$ and (2.5.27) is obvious. Hence, we may assume with no loss of generality that $d(E_k) > d(E)/2 > 0$ and $d(F_k) > d(F)/2 > 0$. If the right-hand side in (2.5.27) is equal to ∞, then (2.5.27) is trivial. Consequently, we may also assume later on that the finite limit is

$$\lim_{k\to\infty} M(\Delta(E_k,F_k;D)) \ \leq\ M_0 \ <\ \infty. \tag{2.5.28}$$

Moreover, we may assume that, for all $k = 1, 2, \ldots,$

$$r_k := h(E, E_k) < r_o := \min\left(\operatorname{dist}(E,F),\ \operatorname{dist}(E,\partial D),\ \operatorname{diam} E/2\right)$$

and

$$r_k^* := h(F, F_k) < r_o^* := \min\left(\operatorname{dist}(E,F),\ \operatorname{dist}(F,\partial D),\ \operatorname{diam} F/2\right).$$

In view of (2.5.28), we may assume that $M(\Gamma_k) < M_o + 1$, where $\Gamma_k = \Delta(E_k, F_k; D)$, $k = 1, 2, \ldots$. Then, for every $k = 1, 2, \ldots$, there is $\rho_k \in \operatorname{adm} \Gamma_k$ such that

$$\int_D \rho_k^2 \, \mathrm{d}m(z) \ < \ M(\Gamma_k) \ + \ \frac{1}{k} \ < \ M: \ = \ M_0 \ + \ 2. \tag{2.5.29}$$

Now, let γ be an arbitrary curve in the family $\Gamma := \Delta(E, F; D)$ and let z_0 be the end of γ in E. Then $\operatorname{dist}(z_0, E_k) \le h(E, E_k)$ and, by the above choice of r_k, every circle $|z - z_0| = r$ with $r \in (r_k, r_0)$ intersects γ as well as E_k and belongs to D. By (2.5.29) and the Jensen inequality, we have

$$\begin{aligned}
M &> \int_{B_0} \rho_k^2 \, \mathrm{d}m(z) \ge \int_{r_k}^{r_0} \left(\int_{|z-z_0|=r} \rho^2 \, |\mathrm{d}z| \right) \mathrm{d}r \\
&= \int_{r_k}^{r_0} \left(⨍_{|z-z_0|=r} \rho^2 \, |\mathrm{d}z| \right) 2\pi r \, \mathrm{d}r \ge \int_{r_k}^{r_0} \left(⨍_{|z-z_0|=r} \rho \, |\mathrm{d}z| \right)^2 2\pi r \, \mathrm{d}r \\
&= \int_{r_k}^{r_0} \left(\int_{|z-z_0|=r} \rho_k \, |\mathrm{d}z| \right)^2 \frac{\mathrm{d}r}{2\pi r} \ \ge \ \frac{1}{2\pi} \log \frac{r_0}{r_k} \inf_{r \in (r_k, r_0)} \left(\int_{|z-z_0|=r} \rho_k \, |\mathrm{d}z| \right)^2,
\end{aligned}$$

where $B_0 = \{z : |z - z_0| < r_0\}$. We obtain from here that there is a circle $|z - z_0| = r$ with $r \in (r_k, r_0)$ that intersects γ and E_k and

$$\int_{|z-z_0|=r} \rho_k \, |\mathrm{d}z| < \delta_k := 2\sqrt{\frac{2\pi M}{\log \frac{r_0}{r_k}}},$$

where $\delta_k \to 0$ as $k \to \infty$. Thus, the curve γ can be joined with E_k by an arc γ_k of a circle such that

$$\int_{\gamma_k} \rho_k \, |\mathrm{d}z| < \delta_k \to 0 \qquad \text{as} \quad k \to \infty.$$

Similarly, the curve γ can be joined with F_k by an arc γ_k^* in D such that

$$\int_{\gamma_k^*} \rho_k \, |\mathrm{d}z| < \delta_k^* \to 0 \qquad \text{as} \quad k \to \infty,$$

where

$$\delta_k^* = 2\sqrt{\frac{2\pi M}{\log \frac{r_0^*}{r_k^*}}}.$$

One can construct a curve $\alpha_k \in \Gamma_k$ consisting of the arcs γ_k, γ_k^*, and pieces of the curve γ such that

$$\int_{\alpha_k} \rho_k \, |\mathrm{d}z| < \int_{\gamma} \rho_k \, |\mathrm{d}z| + \delta_k + \delta_k^*,$$

and then

$$\int_{\gamma} \rho_k \, |\mathrm{d}z| > 1 - \delta_k - \delta_k^*$$

because $\rho_k \in \operatorname{adm} \Gamma_k$. Thus,

$$\rho_k^* : = \frac{\rho_k}{1 - \delta_k - \delta_k *} \in \operatorname{adm} \Gamma$$

because the choice of $\delta_k \to 0$ and $\delta_k^* \to 0$ was independent on $\gamma \in \Gamma$. Hence,

$$M(\Delta(E,F;D)) \leq \frac{\int_D \rho_k^2 \, \mathrm{d}m(z)}{(1 - \delta_k - \delta_k^*)^2}$$

and, finally, by (2.5.29) we obtain (2.5.27). □

Remark 2.6. The main idea for the proof of Lemma 2.15 for extremal lengths is due to Professor P.M. Tamrazov; see Theorem 1 in [251]. In fact, the equality

$$\lim_{k\to\infty} M(\Delta(E_k,F_k;D)) = M(\Delta(E,F;D)) \tag{2.5.30}$$

was even proved there and under more general conditions; cf. also [87, 101] and [270]. However, we just use only (2.5.27).

2.6 Convergence Theorems for Homeomorphisms

The following convergence theorem plays a significant role in our scheme on deriving existence theorems for the Beltrami equation (B); see [222] or [220].

Theorem 2.1. *Let D be a domain in $\mathbb{C}$ and let f_n be a sequence of sense-preserving* ACL *homeomorphisms of D into $\mathbb{C}$ with complex dilatations μ_n such that*

$$\frac{1+|\mu_n(z)|}{1-|\mu_n(z)|} \leq Q(z) \in L^1_{\mathrm{loc}}(D) \qquad \forall\, n = 1,2,\ldots. \tag{2.6.1}$$

If $f_n \to f$ uniformly on each compact set in D, where $f : D \to \mathbb{C}$ is a homeomorphism, then $f \in$ ACL and ∂f_n and $\overline{\partial} f_n$ converge weakly in $L^1_{\mathrm{loc}}(D)$ to ∂f and $\overline{\partial} f$, respectively. Moreover, if in addition $\mu_n \to \mu$ a.e., then $\overline{\partial} f = \mu \partial f$ a.e.

Proof of Theorem 2.1. To prove the first part of the theorem it suffices to show that ∂f_n and $\overline{\partial} f_n$ are uniformly bounded in $L^1_{\mathrm{loc}}(D)$ and have locally absolute equicontinuous indefinite integrals. So, let C be a compact set in D and let V be an open set with their compact closure $\overline{V}$ in D such that $C \subset V$, say $V = \{z \in D : \mathrm{dist}(z,C) < r\}$ where $r < \mathrm{dist}(C, \partial D)$. Note that

$$|\,\overline{\partial} f_n| \le |\,\partial f_n| \le |\,\partial f_n| + |\,\overline{\partial} f_n| \le Q^{1/2}(z) \cdot J_n^{1/2}(z) \qquad \text{a.e.},$$

where J_n is the Jacobian of f_n. Consequently, by the Hölder inequality and Lemma III.3.3 in [152]

$$\int_E |\,\partial f_n|\ \mathrm{d}m(z) \le \left| \int_E Q(z)\ \mathrm{d}m(z) \right|^{1/2} \ |\,f_n(C)\,|^{1/2}$$

for every measurable set $E \subseteq C$. Hence, by the uniform convergence of f_n to f on C,

$$\int_E |\,\partial f_n|\ \mathrm{d}m(z)\ \le \left| \int_E Q(z)\ \mathrm{d}m(z) \right|^{1/2} \ |\,f(V)\,|^{1/2} \tag{2.6.2}$$

for large enough n and, thus, the first part of the proof is complete.

Now, assume that $\mu_n(z) \to \mu(z)$ a.e. Set $\zeta(z) = \overline{\partial} f(z) - \mu(z)\partial f(z)$ and show that $\zeta(z) = 0$ a.e. Indeed, for every disk B with $\overline{B} \subset D$, by the triangle inequality

$$\left| \int_B \zeta(z)\,\mathrm{d}x\,\mathrm{d}y \right| \le I_1(n) + I_2(n),$$

where

$$I_1(n) = \left| \int_B \left(\overline{\partial} f(z) - \overline{\partial} f_n(z) \right) \mathrm{d}x\,\mathrm{d}y \right|$$

and

$$I_2(n) = \left| \int_B (\mu(z)\,\partial f(z) - \mu_n(z)\,\partial f_n(z))\ \mathrm{d}x\,\mathrm{d}y \right|.$$

Note that $I_1(n) \to 0$ because $\overline{\partial} f_n \to \overline{\partial} f$ weakly in L^1_{loc} by the first part of the proof. Next, $I_2(n) = I_2'(n) + I_2''(n)$, where

$$I_2'(n) = \left| \int_B \mu(z)(\partial f(z) - \partial f_n(z))\, \mathrm{d}x\, \mathrm{d}y \right|$$

and

$$I_2''(n) = \left| \int_B (\mu(z) - \mu_n(z))\partial f_n(z)\, \mathrm{d}x\, \mathrm{d}y \right|.$$

Again, by the weak convergence $\partial f_n \to \partial f$ in $L^1_{\mathrm{loc}}(D)$, we have that $I_2'(n) \to 0$ because $\mu \in L^\infty(D)$. Moreover, given $\varepsilon > 0$, by (2.6.2)

$$\int_E |\, \partial f_n(z)|\; \mathrm{d}m(z) < \varepsilon, \quad n = 1, 2, \ldots, \tag{2.6.3}$$

whenever E is every measurable set in B with $|E| < \delta$ for small enough $\delta > 0$.

Further, by the Egoroff theorem (see, e.g., III.6.12 in [73]), $\mu_n(z) \to \mu(z)$ uniformly on some set $S \subset B$ such that $|E| < \delta$, where $E = B \setminus S$. Hence, $|\mu_n(z) - \mu(z)| < \varepsilon$ on S and

$$\begin{aligned} I_2''(n) &\leq \varepsilon \int_S |\, \partial f_n(z)|\; \mathrm{d}m(z) + 2\int_E |\, \partial f_n(z)|\; \mathrm{d}m(z) \\ &\leq \varepsilon \left\{ \left(\int_B Q(z)\; \mathrm{d}m(z) \right)^{1/2} \cdot |\, f(\lambda B)|^{1/2} + 2 \right\} \end{aligned}$$

for some $\lambda > 1$ and for all large enough n, i.e., $I_2''(n) \to 0$, because $\varepsilon > 0$ is arbitrary. Thus, $\int_B \zeta(z)\, \mathrm{d}x\, \mathrm{d}y = 0$ for all disks B with $\overline{B} \subset D$. Finally, by the Lebesgue theorem on the differentiability of an indefinite integral (see, e.g., IV(6.3) in [224]), $\zeta(z) = 0$ a.e. in D. □

Remark 2.7. In fact, under the condition (2.6.1), f_n belongs to $W^{1,1}_{\mathrm{loc}}(D)$ by Proposition 2.4 and f in $W^{1,1}_{\mathrm{loc}}(D)$ by Lemma 2.11; see also (2.6.2). Moreover, if in addition $Q \in L^p_{\mathrm{loc}}(D)$, then f_n belongs to $W^{1,s}_{\mathrm{loc}}(D)$ where $s = 2/(1+1/p)$ by Proposition 2.4. Similar to (2.4.12) and (2.6.2), we have that

$$\|\overline{\partial} f\|_s \;\leq\; \|\partial f\|_s \;\leq\; \|Q\|_p^{1/2} \cdot |f(\lambda B)|^{1/2} \tag{2.6.4}$$

in every ball B such that $\bar{B} \subset D$ and some $\lambda > 1$ and large enough n. Hence, in the case by the known criterion of weak compactness in the spaces L^s, $s \in (1, \infty)$ (see, e.g., Corollary IV.8.4 in [73]), $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^s_{\mathrm{loc}}(D)$. Finally, note that f is a $Q(z)$-qc mapping (see, e.g., [212]), although we no longer need the latter fact.

Proposition 2.6. *Let D be a domain in $\overline{\mathbb{C}}$ and $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots$, a sequence of homeomorphisms such that $f_n \to f$ uniformly on compact sets in D with respect to the spherical (chordal) metric. If the limit function f is discrete, then f is a homeomorphism.*

Proof. Indeed, suppose that $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$ in D. For small $t > 0$, let D_t be a disk of spherical radius t centered at z_1 such that $\overline{D_t} \subset D$ and $z_2 \notin \overline{D_t}$. Then for all n, $f_n(\partial D_t)$ separates $f_n(z_1)$ from $f_n(z_2)$ and, hence, $s(f_n(z_1), f_n(\partial D_t)) < s(f_n(z_1), f_n(z_2))$. Thus, for every such t, there is $\zeta_n(t) \in \partial D_t$ such that $s(f_n(z_1), f_n(\zeta_n(t)) < s(f_n(z_1), f_n(z_2))$. Moreover, there is a subsequence $\zeta_{n_k}(t) \to \zeta_0(t) \in \partial D_t$ because the circle ∂D_t is a compact set. However, the locally uniform convergence $f_{n_k} \to f$ implies that $f_{n_k}(\zeta_{n_k}(t)) \to f(\zeta_0(t))$; see, e.g., [74], p. 268. Consequently, $s(f(z_1), f(\zeta_0(t)) \le s(f(z_1), f(z_2))$. Then, since $f(z_1) = f(z_2)$, there is a point $z_t = \zeta_0(t)$ on ∂D_t such that $f(z_1) = f(z_t)$ for every small t, contradicting the discreteness of f. □

Corollary 2.3. *Let D be a domain in $\mathbb{C}$ and $f_n : D \to \mathbb{C}$, $n = 1, 2, \ldots$, a sequence of quasiconformal mappings which satisfies (2.6.1). If $f_n \to f$ locally uniformly, then either f is constant or f is an* ACL *homeomorphism, and ∂f_n and $\bar{\partial} f_n$ converge weakly in $L^1_{\mathrm{loc}}(D \setminus \{f^{-1}(\infty)\})$ to ∂f and $\bar{\partial} f$, respectively. If in addition $\mu_n \to \mu$ a.e., then $\bar{\partial} f = \mu \partial f$ a.e.*

Proof. Consider the case when f is not constant in D. Let us first show that then no point in D has a neighborhood of the constancy for f. Indeed, assume that there is at least one point $z_0 \in D$ such that $f(z) \equiv c$ for some $c \in \overline{\mathbb{C}}$ in a neighborhood of z_0. Note that the set Ω_0 of such points z_0 is open. The set $E_c = \{z \in D : s(f(z), c) > 0\}$, where s is the spherical (chord) distance in $\overline{\mathbb{C}}$, is also open in view of continuity of f and not empty in the considered case. Thus, there is a point $z_0 \in \partial \Omega_0 \cap D$ because D is connected. By continuity of f, we have that $f(z_0) = c$. However, by the construction, there is a point $z_1 \in E_c = D \setminus \overline{\Omega_0}$ such that $|z_0 - z_1| < r_0 = \mathrm{dist}\,(z_0, \partial D)$, and thus, by the lower estimate of the distance $s(f(z_0), f(z))$ in Lemma 2.12, we obtain a contradiction for $z \in \Omega_0$. Then again, by Lemma 2.12, we obtain that the mapping f is discrete and hence f is a homeomorphism by Proposition 2.6. All other assertions follow from Theorem 2.1. □

The *kernel of a sequence of open sets* $\Omega_n \subseteq \overline{\mathbb{C}}$, $n = 1, 2, \ldots$ is the open set

$$\Omega_0 \;=\; \mathrm{Kern}\; \Omega_n \;:\; = \bigcup_{m=1}^{\infty} \mathrm{Int} \left(\bigcap_{n=m}^{\infty} \Omega_n \right),$$

where $\mathrm{Int}\, A$ denotes the set consisting of all inner points of A; in other words, $\mathrm{Int}\, A$ is the union of all open disks in A with respect to the spherical distance.

Proposition 2.7. *Let $h_n : D \to D'_n$, $D'_n = h_n(D)$, be a sequence of homeomorphisms given in a domain $D \subseteq \overline{\mathbb{C}}$. If h_n converges as $n \to \infty$ locally uniformly with respect to the spherical (chordal) metric to a homeomorphism $h : D \to D' \subseteq \overline{\mathbb{C}}$, then $D' = h(D) \subseteq \operatorname{Kern} D'_n$.*

Proof. Indeed, set $B_s(z_0,\rho) = \{z \in \overline{\mathbb{C}} : s(z,z_0) < \rho\}$, where $z_0 \in D$ and ρ is less than the spherical distance from z_0 to ∂D. Then

$$r_0 \ := \ \min_{z \in \partial B_s(z_0,\rho)} \ s(h(z_0),h(z)) \ > \ 0.$$

Since $h_n \to h$ uniformly on the compact set $\partial B_s(z_0,\rho) = \{z \in \overline{\mathbb{C}} : s(z,z_0) = \rho\}$, there is a large enough integer N such that, for all $n \geq N$, $h_n(z_0) \in B_s(h(z_0),r_0/2)$ and simultaneously

$$B_s(h(z_0),r_0/2) \ \cap \ h_n(\partial B_s(z_0,\rho)) \ = \ B_s(h(z_0),r_0/2) \ \cap \ \partial h_n(B_s(z_0,\rho)) \ = \ \emptyset.$$

Hence, by the connectedness of the disk $B_s(h(z_0),r_0/2)$,

$$B_s(h(z_0),r_0/2) \ \subseteq \ h_n(B_s(z_0,\rho)) \qquad \forall\, n \geq N$$

and, thus, $h(z_0) \in \operatorname{Kern} D'_n$, i.e. $D' \subseteq \operatorname{Kern} D'_n$ by arbitrariness of $\zeta_0 = h(z_0) \in D'$. □

Remark 2.8. In particular, Proposition 2.7 implies that $h(D) \subseteq \mathbb{C}$ if $h_n(D) \subseteq \mathbb{C}$ for all $n = 1,2,\ldots$.

Lemma 2.16. *Let D be a domain in $\overline{\mathbb{C}}$ and let $f_n : D \to \overline{\mathbb{C}}$ be a sequence of homeomorphisms from D into $\overline{\mathbb{C}}$ such that $f_n \to f$ as $n \to \infty$ locally uniformly with respect to the spherical metric to a homeomorphism f from D into $\overline{\mathbb{C}}$. Then $f_n^{-1} \to f^{-1}$ locally uniformly in $f(D)$, too.*

Proof. Set $g_n = f_n^{-1}$ and $g = f^{-1}$. The locally uniform convergence $g_n \to g$ is equivalent to the so-called continuous convergence, meaning that $g_n(u_n) \to g(u_0)$ for every convergent sequence $u_n \to u_0$ in $f(D)$; see [74], p. 268. So, let $u_n \in f(D)$, $n = 0,1,2,\ldots$, and $u_n \to u_0$ as $n \to \infty$. Let us show that $z_n := g(u_n) \to z_0 := g(u_0)$ as $n \to \infty$. It is known that every metric space is $\mathscr{L}^*$ space, i.e., a space with a convergence (see, e.g., Theorem 2.1.1 in [142]), and the Uhryson axiom in the compact spaces says that $z_n \to z_0$ as $n \to \infty$ if and only if, for every convergent subsequence $z_{n_k} \to z_*$, the equality $z_* = z_0$ holds; see, e.g., the Definition 20.1.3 in [142]. Hence, it suffices to prove that the equality $z_* = z_0$ holds for every convergent subsequence $z_{n_k} \to z_*$ as $k \to \infty$. Let D_0 be a subdomain of D such that $z_0 \in D_0$ and $\overline{D}_0$ is a compact subset of D. Then by Proposition 2.7, $f(D_0) \subseteq \operatorname{Kern} f_n(D_0)$ and hence u_0 together with its neighborhood belongs to $f_{n_k}(D_0)$ for all $k \geq K$. Thus, with no loss of generality, we may assume that $u_{n_k} \in f_{n_k}(D_0)$, i.e., $z_{n_k} \in D_0$ for all $k = 1,2,\ldots$, and consequently $z_* \in D$. Then, by the continuous convergence $f_n \to f$, we have that $f_{n_k}(z_{n_k}) \to f(z_*)$, i.e., $f_{n_k}(g_{n_k}(u_{n_k})) = u_{n_k} \to f(z_*)$. The latter implies that $u_0 = f(z_*)$, i.e., $z_* = z_0$. The proof is complete. □

Corollary 2.4. *Let D be a domain in $\mathbb{C}$ and $f_n : D \to \mathbb{C}$, $n = 1, 2, \ldots$, a sequence of quasiconformal mappings which satisfies (2.6.1). If $f_n \to f$ locally uniformly where f is a homeomorphism, then $f^{-1} \in W^{1,2}_{\mathrm{loc}}$.*

Proof. By Lemma 2.16, it follows that $g_n = f_n^{-1} \to f^{-1} = g$ locally uniformly. By Proposition 2.4, f_n and $g_n \in W^{1,2}_{\mathrm{loc}}$ and hence a change of variables further is permitted; see, e.g., Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [152]. Thus (see also formulas I.C(3) in [9]), we obtain that for large n

$$\int\limits_{B} |\partial g_n|^2 \, \mathrm{d}u\mathrm{d}v \; = \int\limits_{g_n(B)} \frac{\mathrm{d}m(z)}{1 - |\mu_n(z)|^2} \; \leq \int\limits_{B^*} Q(z) \, \mathrm{d}m(z) \; < \; \infty, \tag{2.6.5}$$

where B^* and B are relatively compact domains in D and $f(D)$, respectively, such that $g(\bar{B}) \subset B^*$. This estimate implies that the sequence g_n is bounded in $\mathrm{W}^{1,2}(B)$ and, consequently, $f^{-1} \in \mathrm{W}^{1,2}_{\mathrm{loc}}(f(D))$ by Lemma 2.10. □

2.7 Ring Q-Homeomorphisms at Inner Points

Motivated by the ring definition of quasiconformality in [89], the following notion was introduced in [214, 215]: Let D be a domain in $\mathbb{C}$, $z_0 \in D$, and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{C}$ is called a *ring Q-homeomorphism* at the point z_0 if

$$M(\Delta(fC_1, fC_2, fD)) \; \leq \int\limits_{A} Q(z) \cdot \eta^2(|z - z_0|) \, \mathrm{d}m(z) \tag{2.7.1}$$

for every ring

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \quad 0 < r_1 < r_2 < \mathrm{dist}(z_0, \partial D),$$

and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r) \, \mathrm{d}r \; = \; 1 \tag{2.7.2}$$

and where $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$.

Recall a criterion of ring Q-homeomorphisms obtained as Theorem 2.1 in [208]; see also Theorem 7.2 in [165]. Further, we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$; see, e.g., [224], p. 6.

Lemma 2.17. *Let D be a domain in $\mathbb{C}$ and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{C}$ is a ring Q-homeomorphism at a point $z_0 \in D$ if and* *only if, for every* $0 < r_1 < r_2 < d_0 = \operatorname{dist}(z_0, \partial D)$,

$$M(\Delta(fC_1, fC_2, fD)) \leq \frac{2\pi}{I}, \tag{2.7.3}$$

where $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$, $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$, *and*

$$I = I(r_1, r_2) = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r q_{z_0}(r)}, \tag{2.7.4}$$

where $q_{z_0}(r)$ *is the mean value of* $Q(z)$ *over the circle* $|z - z_0| = r$.

Note that the infimum from the right-hand side in (2.7.1) holds for the function

$$\eta_0(r) = \frac{1}{I r q_{z_0}(r)}. \tag{2.7.5}$$

In what follows, we call a homeomorphism $f : D \to \mathbb{C}$ of the class $W^{1,1}_{\text{loc}}$ a *regular homeomorphism* if $J_f(z) > 0$ a.e. Note that every regular homeomorphism satisfies a Beltrami equation (B). Given a point z_0 in $\overline{D}$, the *tangential dilatation* of f as well as the corresponding Beltrami equation (B) with respect to z_0 is the function

$$K^T_\mu(z, z_0) = \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}; \tag{2.7.6}$$

see [214, 215], cf. the corresponding terms and notations in [13, 14, 18, 66, 99, 147] and [198]. For the geometrical sense of the dilatation, see Sect. 6.1; here, note only that

$$K^T_\mu(z, z_0) = \frac{|f_\theta(z)|^2}{r^2 J_f(z)}, \quad \text{where } z = z_0 + r\mathrm{e}^{\mathrm{i}\theta}. \tag{2.7.7}$$

The following theorem was proved in the paper [225]:

Theorem 2.2. *Let* $f : D \to \mathbb{C}$ *be a regular homeomorphism. Then* f *is a ring* Q*-homeomorphisms at a point* $z_0 \in D$ *with* $Q(z) = K^T_\mu(z, z_0)$, $\mu = \mu_f$.

Proof. Without loss of generality, we may assume that $z_0 = 0 \in D$. Consider the ring $R = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Then there is a conformal map h mapping the ring fR onto a ring $R^* = \{w : 1 < |w| < L\}$.

Let Γ^* be the family of paths joining boundary components $|w| = 1$ and $|w| = L$ of the ring R^*. Then, in view of conformal invariants of modulus, $M(\Gamma^*) = M(\Gamma)$, where Γ is the family of all paths joining the boundary components of the ring fR. Thus,

$$M(\Gamma) = \frac{4\pi^2}{\int\limits_{R^*} \frac{\mathrm{d}u\mathrm{d}v}{|w|^2}}.$$

Denote by C_r, $r_1 < r < r_2$ circles $\{z : |z| = r\}$. For $g = h \circ f$, we have that $g \in W^{1,1}_{\mathrm{loc}}(R)$, and hence g is a.e. differentiable and absolutely continuous on C_r for a.e. $r \in (r_1, r_2)$. The latter follows from the invariance of the class $W^{1,1}_{\mathrm{loc}}$ under locally quasi-isometric transformations of coordinates; see, e.g., 1.1.7 in [169]. Note that

$$\int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi} \frac{J_g(re^{i\theta})}{|g(re^{i\theta})|^2}\, r\,dr\,d\theta \le \int\limits_{R^*} \frac{du\,dv}{|w|^2} = \frac{(2\pi)^2}{M(\Gamma)}, \tag{2.7.8}$$

where $w = u + iv$, J_g is the Jacobian of g; see, e.g., Lemma III.3.3 in [152].

Now, we have

$$2\pi \le \int\limits_{C_r} |d\arg g| = \int\limits_{C_r} \frac{|dg(z)|}{|g(z)|} = \int\limits_{0}^{2\pi} \frac{|g_\theta(re^{i\theta})|}{|g(re^{i\theta})|}\, d\theta$$

for a.e. $r \in (r_1, r_2)$, and applying the Schwarz inequality (see, e.g., Theorem I.4 in [29]), we obtain that

$$(2\pi)^2 \le \left(\int\limits_{0}^{2\pi} \frac{|g_\theta(re^{i\theta})|}{|g(re^{i\theta})|}\, d\theta\right)^2 \le \int\limits_{0}^{2\pi} \frac{|g_\theta(re^{i\theta})|^2}{J(re^{i\theta})}\, d\theta \int\limits_{0}^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta\,,$$

i.e.,

$$\frac{2\pi}{r\frac{1}{2\pi}\int_0^{2\pi}\frac{|g_\theta(re^{i\theta})|^2}{r^2 J(re^{i\theta})}\, d\theta} \le r\int\limits_{0}^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta\,. \tag{2.7.9}$$

Setting (see (2.7.7)),

$$k(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|g_\theta(re^{i\theta})|^2}{r^2 J(re^{i\theta})}\, d\theta = \frac{1}{2\pi r}\int\limits_{C_r} K^T_\mu(z, z_0)\,|dz|,$$

and integrating both sides of the inequality (2.7.9) over $r \in (r_1, r_2)$, we see that

$$2\pi\int\limits_{r_1}^{r_2} \frac{dr}{r\,k(r)} \le \int\limits_{r_1}^{r_2} r\,dr \int\limits_{0}^{2\pi} \frac{J(re^{i\theta})}{|g(re^{i\theta})|^2}\, d\theta\,.$$

Combining the last inequality and (2.7.8), we have by the Fubini theorem that

$$\int\limits_{r_1}^{r_2} \frac{dr}{r\,k(r)} \le \frac{2\pi}{M(\Gamma)}\,.$$

Thus,

$$M(\Gamma) \leq \frac{2\pi}{\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{rk(r)}}.$$

Finally, applying Lemma 2.1, we obtain the conclusion of the theorem. □

Corollary 2.5. *Every regular homeomorphism $f : D \to \mathbb{C}$ is a ring Q-homeomorphism with $Q(z) = K_\mu(z)$, $\mu = \mu_f$, at each point $z_0 \in D$.*

Thus, the theory of ring Q-homeomorphisms can be applied to regular homeomorphisms of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ in the plane; see, e.g., Chap. 7 in [165].

2.8 On Some Equivalent Integral Conditions

The main existence theorems for the Beltrami equation (B) are based on integral restrictions on the dilatations $K_\mu(z)$ and $K^T_\mu(z,z_0)$. Here, we establish equivalence of a series of the corresponding integral conditions.

For this goal, we use the following notions of the inverse function for monotone functions: For every nondecreasing function $\Phi : [0,\infty] \to [0,\infty]$, the *inverse function* $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \;=\; \inf_{\Phi(t)\geq\tau} t\,. \tag{2.8.1}$$

Here, inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is nondecreasing too.

Remark 2.9. It is evident immediately by the definition that

$$\Phi^{-1}(\Phi(t)) \;\leq\; t \qquad \forall\, t \in [0,\infty] \tag{2.8.2}$$

with the equality in (2.8.2) except intervals of constancy of the function $\varphi(t)$.

Similarly, for every nonincreasing function $\varphi : [0,\infty] \to [0,\infty]$, we set

$$\varphi^{-1}(\tau) \;=\; \inf_{\varphi(t)\leq\tau} t\,. \tag{2.8.3}$$

Again, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\varphi(t) \leq \tau$ is empty. Note that the function φ^{-1} is also nonincreasing.

Lemma 2.18. *Let $\psi : [0,\infty] \to [0,\infty]$ be a sense-reversing homeomorphism and $\varphi : [0,\infty] \to [0,\infty]$ a monotone function. Then*

$$[\psi\circ\varphi]^{-1}(\tau) \;=\; \varphi^{-1}\circ\psi^{-1}(\tau) \qquad \forall \tau \in [0,\infty] \tag{2.8.4}$$

and

$$[\varphi \circ \psi]^{-1}(\tau) \ \le \ \psi^{-1} \circ \varphi^{-1}(\tau) \qquad \forall \tau \in [0,\infty] \tag{2.8.5}$$

and, except for a countable collection of $\tau \in [0,\infty]$,

$$[\varphi \circ \psi]^{-1}(\tau) \ = \ \psi^{-1} \circ \varphi^{-1}(\tau). \tag{2.8.6}$$

The equality (2.8.6) holds for all $\tau \in [0,\infty]$ *if the function* $\varphi : [0,\infty] \to [0,\infty]$ *is strictly monotone.*

Remark 2.10. If ψ is a sense-preserving homeomorphism, then (2.8.4) and (2.8.6) are obvious for every monotone function φ. Similar notations and statements also hold for other segments $[a,b]$, where a and $b \in [-\infty,+\infty]$, instead of the segment $[0,\infty]$.

Proof of Lemma 2.18. Let us first prove (2.8.4). If φ is nonincreasing, then

$$[\psi \circ \varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t)) \ge \tau} t = \inf_{\varphi(t) \le \psi^{-1}(\tau)} t = \varphi^{-1} \circ \psi^{-1}(\tau).$$

Similarly, if φ is nondecreasing, then

$$[\psi \circ \varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t)) \le \tau} t = \inf_{\varphi(t) \ge \psi^{-1}(\tau)} t = \varphi^{-1} \circ \psi^{-1}(\tau).$$

Now, let us prove (2.8.5) and (2.8.6). If φ is nonincreasing, then applying the substitution $\eta = \psi(t)$, we have

$$[\varphi \circ \psi]^{-1}(\tau) = \inf_{\varphi(\psi(t)) \ge \tau} t = \inf_{\varphi(\eta) \ge \tau} \psi^{-1}(\eta) = \psi^{-1}\left(\sup_{\varphi(\eta) \ge \tau} \eta\right)$$
$$\le \psi^{-1}\left(\inf_{\varphi(\eta) \le \tau} \eta\right) = \psi^{-1} \circ \varphi^{-1}(\tau),$$

i.e., (2.8.5) holds for all $\tau \in [0,\infty]$. It is evident that here strict inequality is possible only for a countable collection of $\tau \in [0,\infty]$ because an interval of constancy of φ corresponds to every such τ. Hence, (2.8.6) holds for all $\tau \in [0,\infty]$ if and only if φ is decreasing.

Similarly, if φ is nondecreasing, then

$$[\varphi \circ \psi]^{-1}(\tau) = \inf_{\varphi(\psi(t)) \le \tau} t = \inf_{\varphi(\eta) \le \tau} \psi^{-1}(\eta) = \psi^{-1}\left(\sup_{\varphi(\eta) \le \tau} \eta\right)$$
$$\le \psi^{-1}\left(\inf_{\varphi(\eta) \ge \tau} \eta\right) = \psi^{-1} \circ \varphi^{-1}(\tau),$$

i.e., (2.8.5) holds for all $\tau \in [0,\infty]$, and again strict inequality is possible only for a countable collection of $\tau \in [0,\infty]$. In that case, (2.8.6) holds for all $\tau \in [0,\infty]$ if and only if φ is increasing. □

Corollary 2.6. *In particular, if* $\varphi : [0,\infty] \to [0,\infty]$ *is a monotone function and* $\psi = j$ *where* $j(t) = 1/t$*, then* $j^{-1} = j$ *and*

$$[j \circ \varphi]^{-1}(\tau) \;=\; \varphi^{-1} \circ j(\tau) \qquad \forall \tau \in [0,\infty], \tag{2.8.7}$$

i.e.,

$$\varphi^{-1}(\tau) \;=\; \Phi^{-1}(1/\tau) \qquad \forall \tau \in [0,\infty], \tag{2.8.8}$$

where $\Phi = 1/\varphi$,

$$[\varphi \circ j]^{-1}(\tau) \;\leq\; j \circ \varphi^{-1}(\tau) \qquad \forall \tau \in [0,\infty], \tag{2.8.9}$$

i.e., the inverse function of $\varphi(1/t)$ *is dominated by* $1/\varphi^{-1}$*, and except for a countable collection of* $\tau \in [0,\infty]$,

$$[\varphi \circ j]^{-1}(\tau) \;=\; j \circ \varphi^{-1}(\tau). \tag{2.8.10}$$

$1/\varphi^{-1}$ *is the inverse function of* $\varphi(1/t)$ *if and only if the function* φ *is strictly monotone.*

Further, the integral in (2.8.13) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.8.12) and (2.8.14)–(2.8.17) as the ordinary Lebesgue integrals. In (2.8.12) and (2.8.13), we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, respectively, $H(t) = \infty$, for all $t \geq T \in [0,\infty)$.

Theorem 2.3. *Let* $\Phi : [0,\infty] \to [0,\infty]$ *be a nondecreasing function and set*

$$H(t) \;=\; \log \Phi(t). \tag{2.8.11}$$

Then the equality

$$\int\limits_{\sigma}^{\infty} H'(t)\, \frac{\mathrm{d}t}{t} \;=\; \infty \tag{2.8.12}$$

implies the equality

$$\int\limits_{\sigma}^{\infty} \frac{\mathrm{d}H(t)}{t} \;=\; \infty \tag{2.8.13}$$

and (2.8.13) is equivalent to

$$\int\limits_{\sigma}^{\infty} H(t)\, \frac{\mathrm{d}t}{t^2} \;=\; \infty \tag{2.8.14}$$

for some $\sigma > 0$, *and (2.8.14) is equivalent to each of the equalities*

$$\int_0^{\delta} H\left(\frac{1}{t}\right)\,\mathrm{d}t \;=\; \infty \tag{2.8.15}$$

for some $\delta > 0$,

$$\int_{\sigma_*}^{\infty} \frac{\mathrm{d}\eta}{H^{-1}(\eta)} \;=\; \infty \tag{2.8.16}$$

for some $\sigma_* > H(+0)$,

$$\int_{\delta_*}^{\infty} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)} \;=\; \infty \tag{2.8.17}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.8.12) is equivalent to (2.8.13) and hence (2.8.12)–(2.8.17) are equivalent to each other if Φ *is in addition absolutely continuous. In particular, all the conditions (2.8.12)–(2.8.17) are equivalent if* Φ *is convex and nondecreasing.*

Remark 2.11. It is necessary to give one more explanation. From the right-hand sides in the conditions (2.8.12)–(2.8.17), we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0,t_*]$, then $H(t) = -\infty$ for $t \in [0,t_*]$, and we complete the definition $H'(t) = 0$ for $t \in [0,t_*]$. Note, the conditions (2.8.13) and (2.8.14) exclude that t_* belongs to the interval of integrability because in the contrary case, the left-hand sides in (2.8.13) and (2.8.14) are either equal to $-\infty$ or indeterminate. Hence, we may assume in (2.8.12)–(2.8.15) that $\sigma > t_0$ where $t_0 :\, = \sup_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$, and $\delta < 1/t_0$, respectively.

Proof. The equality (2.8.12) implies (2.8.13) because, with the exception of the mentioned special case,

$$\int_{\sigma}^{T} \mathrm{d}\Psi(t) \;\geq\; \int_{\sigma}^{T} \Psi'(t)\,\mathrm{d}t \qquad \forall\, T \in (\Delta,\, \infty),$$

where

$$\Psi(t) \,:\, = \int_{\sigma}^{t} \frac{\mathrm{d}H(\tau)}{\tau}, \quad \Psi'(t) \;=\; \frac{H'(t)}{t}$$

(see, e.g., Theorem 7.4 of Chap. IV in [224], p. 119), and hence,

$$\int_{\sigma}^{T} \frac{\mathrm{d}H(t)}{t} \;\geq\; \int_{\sigma}^{T} H'(t)\,\frac{\mathrm{d}t}{t} \qquad \forall\, T \in (\Delta,\, \infty).$$

The equality (2.8.13) is equivalent to (2.8.14) by integration by parts (see, e.g., Theorem III.14.1 in [224], p. 102). Indeed, again except for the mentioned special case, through integration by parts, we have

$$\int_{\sigma}^{T} \frac{\mathrm{d}H(t)}{t} - \int_{\sigma}^{T} H(t)\,\frac{\mathrm{d}t}{t^2} = \frac{H(T+0)}{T} - \frac{H(\sigma-0)}{\sigma} \qquad \forall\, T \in (\sigma,\, \infty)$$

and, if

$$\liminf_{t\to\infty} \frac{H(t)}{t} < \infty,$$

then the equivalence of (2.8.13) and (2.8.14) is obvious. If

$$\lim_{t\to\infty} \frac{H(t)}{t} = \infty,$$

then (2.8.14) obviously holds, $\frac{H(t)}{t} \geq 1$ for $t > t_0$ and

$$\int_{t_0}^{T} \frac{\mathrm{d}H(t)}{t} = \int_{t_0}^{T} \frac{H(t)}{t}\,\frac{\mathrm{d}H(t)}{H(t)} \geq \log\,\frac{H(T)}{H(t_0)} = \log\,\frac{H(T)}{T} + \log\,\frac{T}{H(t_0)} \to \infty$$

as $T \to \infty$, i.e., (2.8.13), holds too.

Now, (2.8.14) is equivalent to (2.8.15) by the change of variables $t \to 1/t$.

Next, (2.8.15) is equivalent to (2.8.16) because by the geometric sense of integrals as areas under graphs of the corresponding integrands,

$$\int_{0}^{\delta} \Psi(t)\,\mathrm{d}t = \int_{\Psi(\delta)}^{\infty} \Psi^{-1}(\eta)\,\mathrm{d}\eta + \delta\cdot\Psi(\delta),$$

where $\Psi(t) = H(1/t)$, and because by Corollary 2.6 the inverse function for $H(1/t)$ coincides with $1/H^{-1}$ at all points except a countable collection.

Further, set $\psi(\xi) = \log\xi$. Then $H = \psi\circ\Phi$ and by Lemma 2.18 and Remark 2.10 $H^{-1} = \Phi^{-1}\circ\psi^{-1}$, i.e., $H^{-1}(\eta) = \Phi^{-1}(\mathrm{e}^{\eta})$, and by the substitutions $\tau = \mathrm{e}^{\eta}$, $\eta = \log\,\tau$, we have the equivalence of (2.8.16) and (2.8.17).

Finally, (2.8.12) and (2.8.13) are equivalent if Φ is absolutely continuous; see, e.g., Theorem IV.7.4 in [224], p. 119. □

2.9 One More Integral Condition

In this section, we establish a useful connection of the integral conditions from the last section with one more condition.

Recall that a function $\psi : [0,\infty] \to [0,\infty]$ is called *convex* if $\psi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ for all t_1 and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

In what follows, $\mathbb{D}$ denotes the unit disk in the plane $\mathbb{C}$,

$$\mathbb{D} \ = \ \{\, z \in \mathbb{C} : \ |z| \ < \ 1 \,\}. \tag{2.9.1}$$

Lemma 2.19. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function and let $\Phi : [0,\infty] \to [0,\infty]$ be a nondecreasing convex function. Then*

$$\int\limits_0^1 \frac{\mathrm{d}r}{rq(r)} \ \ge \ \frac{1}{2} \int\limits_N^\infty \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}, \tag{2.9.2}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$ and

$$N \ = \ \int\limits_{\mathbb{D}} \Phi(Q(z))\ \mathrm{d}m(z). \tag{2.9.3}$$

The more complicated version of the estimate (2.9.2) can be found in Lemma 6.2.

Proof. Note that the result is obvious if $N = \infty$. Hence, we assume further that $N < \infty$. Consequently, we may also assume that $\Phi(t) < \infty$ for all $t \in [0,\infty)$ because in the contrary case, $Q \in L^\infty(\mathbb{D})$, and then the left-hand side in (2.9.2) is equal to ∞. Moreover, we may assume that $\Phi(t)$ is not constant (because in the contrary case $\Phi^{-1}(\tau) \equiv \infty$ for all $\tau > \tau_0$ and hence the right-hand side in (2.9.2) is equal to 0), $\Phi(t)$ is (strictly) increasing, convex, and continuous in a segment $[t_*,\infty]$ for some $t_* \in [0,\infty)$ and

$$\Phi(t) \ \equiv \ \tau_0 \ = \ \Phi(0) \qquad \forall\, t \ \in [0,t_*]. \tag{2.9.4}$$

Next, setting

$$H(t) \ : \ = \ \log\ \Phi(t), \tag{2.9.5}$$

we see by Proposition 2.18 and Remark 2.10 that

$$H^{-1}(\eta) \ = \ \Phi^{-1}(\mathrm{e}^\eta)\,, \quad \Phi^{-1}(\tau) \ = \ H^{-1}(\log\ \tau). \tag{2.9.6}$$

Thus, we obtain that

$$q(r) = H^{-1}\left(\log\frac{h(r)}{r^2}\right) = H^{-1}\left(2\log\frac{1}{r} + \log\ h(r)\right) \qquad \forall\, r \ \in R_*, \tag{2.9.7}$$

where $h(r) \ : \ = \ r^2\Phi(q(r))$ and $R_* \ = \ \{r \in (0,1) : \ q(r) \ > \ t_*\}$. Then also

$$q(\mathrm{e}^{-s}) \ = \ H^{-1}\left(2s \ + \ \log\ h(\mathrm{e}^{-s})\right) \qquad \forall\, s \ \in S_*, \tag{2.9.8}$$

where $S_* \ = \ \{s \in (0,\infty) : \ q(\mathrm{e}^{-s}) \ > \ t_*\}$.

Now, by the Jensen inequality,

$$\int_0^\infty h(\mathrm{e}^{-s})\,\mathrm{d}s = \int_0^1 h(r)\,\frac{\mathrm{d}r}{r} = \int_0^1 \Phi(q(r))\,r\mathrm{d}r$$
$$\leq \int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \Phi(Q(r\mathrm{e}^{\mathrm{i}\vartheta}))\,\mathrm{d}\vartheta\right) r\mathrm{d}r = \frac{N}{2\pi}, \tag{2.9.9}$$

and then

$$|T| = \int_T \mathrm{d}s \leq \frac{1}{2}, \tag{2.9.10}$$

where $T = \{ s \in (0,\infty) : \; h(\mathrm{e}^{-s}) > N/\pi \}$. Let us show that

$$q(\mathrm{e}^{-s}) \leq H^{-1}\left(2s + \log\frac{N}{\pi}\right) \qquad \forall\, s \in (0,\infty)\setminus T_*, \tag{2.9.11}$$

where $T_* = T \cap S_*$, $(0,\infty)\setminus T_* = [(0,\infty)\setminus S_*] \cup [(0,\infty)\setminus T] = [(0,\infty)\setminus S_*] \cup [S_* \setminus T]$. The inequality (2.9.11) holds for $s \in S_* \setminus T$ by (2.9.8) because H^{-1} is a nondecreasing function. Note also that by (2.9.4),

$$\mathrm{e}^{2s}\frac{N}{\pi} = \mathrm{e}^{2s} \fint_{\mathbb{D}} \Phi(Q(z))\,\mathrm{d}m(z) > \Phi(0) = \tau_0 \qquad \forall\, s \in (0,\infty). \tag{2.9.12}$$

Hence, since the function Φ^{-1} is nondecreasing and $\Phi^{-1}(\tau_0 + 0) = t_*$, we have by (2.9.6) that

$$t_* < \Phi^{-1}\left(\frac{N}{\pi}\,\mathrm{e}^{2s}\right) = H^{-1}\left(2s + \log\frac{N}{\pi}\right) \qquad \forall\, s \in (0,\infty). \tag{2.9.13}$$

Consequently, (2.9.11) holds for $s \in (0,\infty)\setminus S_*$ too. Thus, (2.9.11) is true.

Since H^{-1} is nondecreasing, we have by (2.9.10) and (2.9.11) that

$$\int_0^1 \frac{\mathrm{d}r}{rq(r)} = \int_0^\infty \frac{\mathrm{d}s}{q(\mathrm{e}^{-s})} \geq \int_{(0,\infty)\setminus T_*} \frac{\mathrm{d}s}{H^{-1}(2s+\Delta)}$$
$$\geq \int_{|T_*|}^\infty \frac{\mathrm{d}s}{H^{-1}(2s+\Delta)} \geq \int_{\frac{1}{2}}^\infty \frac{\mathrm{d}s}{H^{-1}(2s+\Delta)} = \frac{1}{2}\int_{1+\Delta}^\infty \frac{\mathrm{d}\eta}{H^{-1}(\eta)}, \tag{2.9.14}$$

where $\Delta = \log N/\pi$. Note that $1+\Delta = \log\ N + \log\ e/\pi\ < \log\ N$. Thus,

$$\int\limits_0^1 \frac{\mathrm{d}r}{rq(r)} \geq \frac{1}{2}\int\limits_{\log N}^{\infty} \frac{\mathrm{d}\eta}{H^{-1}(\eta)} \tag{2.9.15}$$

and, after the replacement $\eta = \log\ \tau$, we obtain (2.9.2). ☐

Theorem 2.4. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\ \mathrm{d}m(z)\ <\ \infty, \tag{2.9.16}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a nondecreasing convex function such that

$$\int\limits_{\delta_0}^{\infty} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}\ =\ \infty, \tag{2.9.17}$$

for some $\delta_0\ >\ \tau_0\ :\ =\ \Phi(0)$. Then

$$\int\limits_0^1 \frac{\mathrm{d}r}{rq(r)}\ =\ \infty, \tag{2.9.18}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Remark 2.12. Note that (2.9.17) implies that

$$\int\limits_{\delta}^{\infty} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}\ =\ \infty \tag{2.9.19}$$

for every $\delta\ \in\ [0,\infty)$, but (2.9.19) for some $\delta \in [0,\infty)$, generally speaking, does not imply (2.9.17). Indeed, for $\delta \in [0,\delta_0)$, (2.9.17) evidently implies (2.9.19) and, for $\delta \in (\delta_0,\infty)$, we have that

$$0\ <\ \int\limits_{\delta_0}^{\delta} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}\ \leq\ \frac{1}{\Phi^{-1}(\delta_0)}\ \log\ \frac{\delta}{\delta_0}\ <\ \infty \tag{2.9.20}$$

because Φ^{-1} is nondecreasing and $\Phi^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi^{-1}(\tau) \equiv 0$ for all $\tau \in [0,\tau_0]$, $\tau_0 = \Phi(0)$, and hence (2.9.19) for $\delta \in [0,\tau_0)$, generally speaking, does not imply (2.9.17). If $\tau_0 > 0$, then

$$\int\limits_{\delta}^{\tau_0} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}\ =\ \infty \qquad \forall\ \delta\ \in\ [0,\tau_0). \tag{2.9.21}$$

However, (2.9.21) gives no information on the function $Q(z)$ itself and, consequently, (2.9.19) for $\delta < \Phi(0)$ cannot imply (2.9.18) at all.

By (2.9.19), the proof of Theorem 2.4 is reduced to Lemma 2.19. Combining Theorems 2.3 and 2.4, we also obtain the following conclusion:

Corollary 2.7. *If $\Phi : [0,\infty] \to [0,\infty]$ is a nondecreasing convex function and Q satisfies the condition (2.9.16), then each of the conditions (2.8.12)–(2.8.17) implies (2.9.18).*

2.10 On Weakly Flat and Strongly Accessible Boundaries

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{C}$, defined first in [131] (see also [132]), are localizations and generalizations of the corresponding notions introduced in [162–164]; cf. with the properties P_1 and P_2 by Väisälä in [258] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [178]. The significance of such a type of domain is that conformal and quasiconformal mappings as well as many of their generalizations between them admit continuous and homeomorphic extensions to the boundary, respectively. Lemma 2.20 further establishes the relation of weak flatness formulated in terms of moduli of path families with the general topological notion of local connectedness on the boundary; see, e.g., [131] and [207].

Recall that a domain $D \subset \mathbb{C}$ is said to be *locally connected at a point* $z_0 \in \partial D$ if, for every neighborhood U of the point z_0, there is a neighborhood $V \subseteq U$ of z_0 such that $V \cap D$ is connected (in other words, for every ball $B_0 = B(z_0, r_0)$, there is a component of connectivity of $B_0 \cap D$ which includes $B \cap D$ where $B = B(z_0, r)$ for some $r \in (0, r_0)$), see Fig. 2. Note that every Jordan domain D in $\mathbb{C}$ is locally connected at each point of ∂D; see, e.g., [268], p. 66.

The boundary ∂D is called *weakly flat at a point* $z_0 \in \partial D$ if, for every neighborhood U of the point z_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of z_0 such that

$$M(\Delta(E,F;D)) \geq P \tag{2.10.1}$$

for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *weakly flat* if it is weakly flat at every point in ∂D, see Fig. 3.

A point $z_0 \in \partial D$ is also called *strongly accessible* if, for every neighborhood U of the point z_0, there exists a compactum E, a neighborhood $V \subset U$ of z_0 and a number $\delta > 0$ such that

$$M(\Delta(E,F;D)) \geq \delta \tag{2.10.2}$$

for all continua F in D intersecting ∂U and ∂V. It is said that the boundary ∂D is *strongly accessible* if every point $z_0 \in \partial D$ is strongly accessible.

Fig. 2 Local connectedness

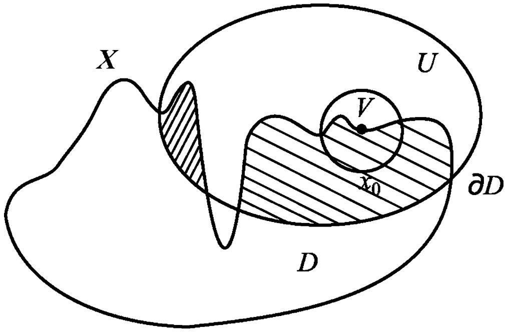
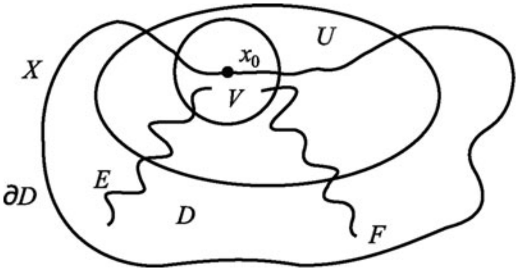


Fig. 3 Weak flatness



Note that in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods U and V of a point z_0 only balls (closed or open) centered at z_0 or only neighborhoods of z_0 in another fundamental system of its neighborhoods. These conceptions can also in a natural way be extended to the case of $\overline{\mathbb{C}}$ and $z_0 = \infty$. Then we must use the corresponding neighborhoods of ∞.

Proposition 2.8. *If a domain D in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then the point z_0 is strongly accessible from D.*

Proof. Indeed, let $U = B(z_0, r_0)$ where $0 < r_0 < d_0 = \sup_{z\in D} |z - z_0|$ and $P_0 \in (0, \infty)$. Then by the condition, there is $r \in (0, r_0)$ such that

$$M(\Delta(E,F;D)) \geq P_0 \tag{2.10.3}$$

for all continua E and F in D intersecting $\partial B(z_0, r_0)$ and $\partial B(z_0, r)$. Choose an arbitrary path connecting $\partial B(z_0, r_0)$ and $\partial B(z_0, r)$ in D as a compactum E. Then, for every continuum F in D intersecting $\partial B(z_0, r_0)$ and $\partial B(z_0, r)$, the inequality (2.10.3) holds. □

Corollary 2.8. *Weakly flat boundaries of domains in $\mathbb{C}$ are strongly accessible.*

Lemma 2.20. *If a domain D in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then D is locally connected at z_0.*

Proof. Indeed, let us assume that the domain D is not locally connected at the point z_0. Then there is a positive number $r_0 < d_0 = \sup_{z\in D}|z-z_0|$ such that, for every neighborhood $V \subseteq U := B(z_0, r_0)$ of z_0, one of the following two conditions holds:

a) $V \cap D$ has at least two connected components K_1 and K_2 with $z_0 \in \overline{K_1} \cap \overline{K_2}$.
b) $V \cap D$ has a sequence of connected components $K_1, K_2, \ldots, K_m, \ldots$ such that $z_m \to z_0$ as $m \to \infty$ for some $z_m \in K_m$. Note that $\overline{K_m} \cap \partial V \neq \emptyset$ for all $m = 1, 2, \ldots$ in view of the connectivity of D.

In particular, this is true for the neighborhood $V = U = B(z_0, r_0)$. Let r_* be an arbitrary number in the interval $(0, r_0)$. Then, for all $i \neq j$,

$$M(\Delta(K_i^*, K_j^*; D)) \ \leq \ M_0 \ := \ \frac{|D \cap B(z_0, r_0)|}{[2(r_0 - r_*)]^n} \ < \ \infty, \tag{2.10.4}$$

where $K_i^* = K_i \cap \overline{B(z_0, r_*)}$ and $K_j^* = K_j \cap \overline{B(z_0, r_*)}$. Note that the following function is admissible for the path family $\Gamma_{ij} = \Delta(K_i^*, K_j^*; D)$:

$$\rho(z) \ = \ \begin{cases} \frac{1}{2(r_0 - r_*)} & \text{for } \ z \in B_0 \setminus \overline{B_*}, \\ 0 & \text{for } \ z \in \mathbb{C} \setminus (B_0 \setminus \overline{B_*}), \end{cases}$$

where $B_0 = B(z_0, r_0)$ and $B_* = B(z_0, r_*)$ because K_i and K_j as components of connectivity for $D \cap B_0$ cannot be connected by a path in B_0 and hence every path connecting K_i^* and K_j^* must go through the ring $B_0 \setminus \overline{B_*}$ at least twice.

However, in view of a) and b), we obtain a contradiction between (2.10.4) and the condition of weak flatness of ∂D at z_0. Indeed, by the condition, there is $r \in (0, r_*)$ such that

$$M(\Delta(E, F; D)) \ \geq \ 2M_0 \tag{2.10.5}$$

for all continua E and F in D intersecting the spheres $|z - z_0| = r_*$ and $|z - z_0| = r$. By a) and b) there is a pair of components K_{i_0} and K_{j_0} of $D \cap B_0$ which intersect both spheres. Let us choose points $z_0 \in K_{i_0} \cap B$ and $\zeta_0 \in K_{j_0} \cap B$ where $B = B(z_0, r)$ and connect them by a path C in D. Let C_1 and C_2 be the components of $C \cap K_{i_0}^*$ and $C \cap K_{j_0}^*$, including the points z_0 and ζ_0, respectively. Then by (2.10.4),

$$M(\Delta(C_1, C_2; D)) \ \leq \ M_0,$$

and by (2.10.5),

$$M(\Delta(C_1, C_2; D)) \ \geq \ 2M_0.$$

The contradiction disproves the assumption that D is not locally connected at z_0. □

Corollary 2.9. *A domain D in $\mathbb{C}$ with a weakly flat boundary is locally connected at each of its boundary points.*

We show further that all theorems on a homeomorphic extension to the boundary of quasiconformal mappings are valid for the so-called strong ring solutions of the Beltrami equation (B) under the condition of weak flatness of boundaries (2.10.1); see Chaps. 7 and 8. The condition of strong accessibility (2.10.2) plays a similar role for a continuous extension of such solutions to the boundary.

Chapter 3
The Classical Beltrami Equation $||\mu||_\infty < 1$

3.1 Quasiconformal Mappings

Consider the Beltrami equation (B) in Sect. 1.1 with $||\mu||_\infty < 1$, and let $f : D \to \mathbb{C}$ be its homeomorphic solution. Since $|\mu| < 1$ a.e., f is sense preserving, and since $||\mu||_\infty < 1$, f is a quasiconformal mapping.

There are several equivalent definitions of quasiconformality and some of them have been extended to mappings between metric spaces; see, e.g., [104] and [102]. For an extension to $\mathbb{R}^n$, see [89, 90, 145, 258, 259], and to Banach spaces, see [260, 261]. According to the analytic definition, a sense-preserving embedding $f : D \to \mathbb{C}$ is *quasiconformal* if f is ACL and there exists $k \in [0,1)$ such that a.e. in D:

$$|\overline{\partial} f(z)| \leq k|\partial f(z)|. \tag{3.1.1}$$

At points $z \in D$ where f is differentiable,

$$K(z) = K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} = \lim_{r\to 0} \frac{\max\limits_{|w-z|=r} |f(z)-f(w)|}{\min\limits_{|w-z|=r} |f(z)-f(w)|} \tag{3.1.2}$$

is the *dilatation* of f at z. Then the *maximal dilatation* of f in D which is defined by

$$K_f = \operatorname*{ess\,sup}_{z\in D} K(z) \tag{3.1.3}$$

satisfies:

$$K_f = \frac{1+||\mu||_\infty}{1-||\mu||_\infty}. \tag{3.1.4}$$

The other two basic types of definitions of quasiconformality are geometric, which use the quasi-invariance of certain conformal invariants like the modulus

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of path families (see, e.g., [5, 152, 188] and [258]), and a metric definition, which is based, roughly speaking, on the extent of roundedness of the image of balls (see, e.g., [104, 258]). If $f : D \to \mathbb{C}$ is a homeomorphic solution of (B) and h is holomorphic in $f(D)$, then $g = h \circ f$ is a solution. Indeed, g is ACL, and ∂g and $\overline{\partial} g$ satisfy $\overline{\partial} g = \mu(z)\partial g$ a.e. in D. Note that since nonconstant holomorphic mappings are open and discrete, so is g. By *Stoilow's factorization theorem* [240], every open and discrete mapping $g : D \to \mathbb{C}$ can be written as a composition $h \circ f$, where f is a homeomorphism and h is a nonconstant holomorphic mapping in $f(D)$. In view of Stoilow's factorization theorem, it is not hard to see that if g is an elementary solution (see Sect. 1.5), then $g = h \circ f$, where f is a homeomorphic solution of (B) and h is holomorphic.

3.2 The Main Problems

The main problems in the classical case concerning solutions are:

- Existence
- Uniqueness
- Regularity
- Representation of the solution set
- Removability of isolated singularities
- Boundary behavior
- Mapping properties

3.3 Integrability

Let $f : D \to \mathbb{C}$ be a homeomorphic solution of (B). Then by using the local integrability of the Jacobian J of f in D and the dilatation inequality,

$$(|\partial f(z)| + |\overline{\partial} f(z)|)^2 \leq \frac{1 + |\mu(z)|}{1 - |\mu(z)|}\, J(z), \tag{3.3.1}$$

which can be easily verified, one can conclude that f belongs to the Sobolev class $W^{1,2}_{\text{loc}}(D)$ and that the partial derivatives ∂f and $\overline{\partial} f$ are a.e. the generalized $W^{1,2}_{\text{loc}}$ derivatives of f. In particular, every homeomorphic solution belongs to $W^{1,1}_{\text{loc}}(D)$.

Bojarski [44] showed that in fact $f \in W^{1,p}_{\text{loc}}(D)$ for some $p > 2$ which depends on $k = ||\mu||_\infty$ or equivalently on K. Settling a long-standing conjecture by Gehring and Reich [93], Astala [22] proved that if $f : D \to \mathbb{C}$ is qc and E is a Borel set in D, then

$$|f(E)| \leq M|E|^{1/K}$$

with a constant $M = M(K) = 1 + O(K-1)$. As a consequence, Astala settled a long-standing conjecture by Gehring and Lehto, proving that $f \in W^{1,p}_{\mathrm{loc}}$ for every $p < p_0$ where

$$p_0 = \frac{2K}{K-1} = 1 + \frac{1}{k} \tag{3.3.2}$$

and p_0 is best possible; see [23].

It should be noted that a $W^{1,2}_{\mathrm{loc}}$ mapping which satisfies the Beltrami equation with $||\mu||_\infty \le k$ is a *K-quasiregular mapping* with $K = (1+k)/(1-k)$, according to the notation in [203], or a *mapping of bounded distortion*, according to the notation in [202]. Note that such solutions are elementary in our notation; see the definition in 1.5.

Astala [23] proved that every $W^{1,q}_{\mathrm{loc}}$ solution of the Beltrami equation (B) with

$$q > q_0 = 1 + k \tag{3.3.3}$$

belongs to $W^{1,2}_{\mathrm{loc}}$ and hence is elementary. Iwaniec and Martin [116] showed that q_0 is the best lower bound, i.e., given $k \in [0,1)$, there is a measurable $\mu : \mathbb{C} \to \mathbb{C}$ with $||\mu||_\infty = k$ such that the (B) has a nonelementary solution f which belongs to $W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ for every $q < q_0$, and does not belong to $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

3.4 The Classical Existence and Uniqueness Theorem

One of the main theorems in the theory of Beltrami equations in the classical case, and hence in the theory of plane quasiconformal mappings, is the following existence and uniqueness theorem:

Theorem 3.1. *Let $\mu(z)$ be a complex valued measurable function in a domain D in the complex plane $\mathbb{C}$ with $||\mu||_\infty < 1$. Then:*

(i) Equation (B) *has a sense-preserving homeomorphic solution $f : D \to \mathbb{C}$.*
(ii) f is unique up to a postcomposition by a conformal mapping.

The following corollary can be derived from Theorem 3.1 and Stoilow's factorization theorem. The corollary says that every homeomorphic solution of (B) in the case $||\mu||_\infty < 1$ generates the set of all elementary solutions.

Corollary 3.1. *Let $h : D \to \mathbb{C}$ be a homeomorphic solution of* (B) *with $||\mu||_\infty < 1$ and f an elementary solution of* (B). *Then there exists a unique holomorphic function g in $h(D)$ such that $f = g \circ h$.*

In view of the corollary, the homeomorphic solutions may be viewed as prime solutions.

Remark 3.1. Given μ as in Theorem 3.1, the measurable metric $\mathrm{d}s = |\mathrm{d}z + \mu\,\mathrm{d}\bar{z}|$ defines a complex structure in D and makes D a Riemann surface thus, every homeomorphic solution of (B) becomes a conformal mapping from D with this conformal structure. For this reason, Theorem 2.11 is known as the measurable Riemann mapping theorem; see [7].

3.5 Methods of Proof of Uniqueness and Existence

In this section, we survey some of the methods in the proofs of existence and uniqueness of homeomorphic solutions.

3.5.1 Uniqueness

Let f and g be two homeomorphic solutions in D. Then f and g are in $W^{1,2}_{\text{loc}}$, and hence, $h = g \circ f^{-1} \in W^{1,1}_{\text{loc}}$, and since the partial derivatives of h satisfy the Cauchy–Riemann equations, h is conformal by Weyl's lemma (see [9]), and the uniqueness follows.

3.5.2 Existence

3.5.2.1 Local Solutions

The standard approach in finding a local solution at a point $z_0 \in D$ goes as follows; cf. [152]. One approximates μ in a certain neighborhood U of z_0 by a sequence of "nice" functions μ_n, like polynomials in z and $\bar{z}$. For each μ_n, one finds a normalized homeomorphic solution $f_n : U_n \to \mathbb{C}, U_n \subset U$, say by substituting a power series in z and $\bar{z}$, where one has much freedom in choosing some of the coefficients. Then one verifies that $V = \cap U_n$ is a neighborhood of z_0. Next, with the aid of suitable distortion theorems for quasiconformal mappings, equicontinuity of the sequence f_n is obtained, and one deduces that there is a subsequence which converges locally uniformly to a homeomorphic solution f of (B) in V. A similar method can be applied in the relaxed classical case.

3.5.2.2 Global Solutions via Local Solutions

In view of Theorem 3.1(ii) (see also Remark 3.1), it suffices to find a local solution of (B) at every point $z \in D$. Then the existence of a global homeomorphic solution follows by the uniformization theorem. Indeed, since every homeomorphic ACL solution is in $W^{1,2}_{\text{loc}}$, it follows that if f and g are two local homeomorphic solutions of, say in subdomains U and V, respectively, then $g \circ f^{-1}$ is conformal in $f(U \cap V)$. It thus follows that the set of all local homeomorphic solutions defines a conformal structure in D, which makes D a planar Riemann surface. By the classical uniformization theorem (cf. [80]), there is a conformal mapping F of D (endowed with the conformal structure which is defined by the local solutions of (B)) into $\mathbb{C}$, meaning if $f : U \to \mathbb{C}, U \subset D$, is a local homeomorphic solution of (B), then $h = F \circ f^{-1}$ is conformal in $f(U)$, and $F|U = h \circ f$. Since a postcomposition of a solution by a conformal mapping is again a solution, it follows that F is a global solution of (B) in D.

3.5.2.3 Global Solutions Directly

There are two basic methods of finding global homeomorphic solutions directly. One of them is based on PDE methods and the other one on singular integral methods.

3.5.2.4 PDE Methods

Following [176], consider the system (B) first for smooth coefficients. Define a boundary value problem for $u(z) = \mathrm{Re} f(z)$ and get the existence of u and then of v. Finally, by using approximation, one gets a global homeomorphic solution in $W^{1,2}_{\mathrm{loc}}$ for the original μ.

3.5.2.5 Singular Integral Methods

Here, we follow [9], [44], and [263]. Consider (B) with a measurable μ in a domain D in $\mathbb{C}$ with $||\mu||_\infty = k < 1$. Extend μ on $\mathbb{C}$ by setting $\mu = 0$ on $\mathbb{C} \setminus D$. Then the extended function, denoted again by μ, is measurable in $\mathbb{C}$ and has the same L^∞ norm k. Obviously, the existence of a homeomorphic solution of (B) in $\mathbb{C}$ yields a homeomorphic solution in D. Therefore, it suffices to consider (B) in $\mathbb{C}$.

We first consider the case where $\mu \in C^1_0(D)$, i.e., μ has continuous partial derivatives and a compact support in $\mathbb{C}$. A homeomorphic solution $f : \mathbb{C} \to \mathbb{C}$ in this case is conformal outside of a certain compact and hence extends to a homeomorphism of $\overline{\mathbb{C}}$ onto itself with $f(\infty) = \infty$. Since a composition of a solution with a function of a form $z \to az + b, a, b \in \mathbb{C}$ is again a solution, we may look for a solution $f : \mathbb{C} \to \mathbb{C}$ which has the Laurent expansion:

$$f(z) = z + g(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \tag{3.5.1}$$

in a neighborhood of ∞.

Setting (3.5.1) in (B), we get

$$g_{\bar z} = \mu(z)[1+g_z]. \tag{3.5.2}$$

Then $g_z = f_z - 1$ is in the class $L^p(\mathbb{C})$ for every $p \geq 1$. In each of these classes, the inverse of the operator $\overline{\partial} = \frac{\partial}{\partial \bar z}$ is given by the Cauchy transform:

$$P : L^p \to L^p, \tag{3.5.3}$$

which is defined by

$$Ph(z) = -\frac{1}{\pi}\int \frac{h(\zeta)}{\zeta - z}\,\mathrm{d}m(\zeta) \tag{3.5.4}$$

where the singular integral is defined over $\mathbb{C}$ as a principal value integral.

By applying P to (3.5.2), we obtain

$$g = Pg_{\bar z} = P\mu(1+g_z) \tag{3.5.5}$$

and hence

$$g_z = T\mu(1+g_z), \tag{3.5.6}$$

where

$$T = \frac{\partial}{\partial z}P = -\frac{1}{\pi}\int \frac{h(\zeta)}{(\zeta - z)^2}\,\mathrm{d}m(\zeta). \tag{3.5.7}$$

Thus, $h := g_z$ satisfies the equation:

$$h = T\mu(1+h). \tag{3.5.8}$$

Since $||T||_p \to 1$ as $p \to 2$, it follows that there is p_0 such that for $2 < p < p_0$,

$$||T\mu||_p = ||T||_p||\mu||_\infty < 1,$$

and hence, by the fixed point theorem for contracting operators, (3.5.8) has a solution $h \in L^p$ which can be found by successive approximations. Thus,

$$f = z + P\mu(1+h) \tag{3.5.9}$$

is a solution of (B).

Next, one shows that f is a local homeomorphism in $\overline{\mathbb{C}}$ and hence a homeomorphism of $\overline{\mathbb{C}}$ onto itself; see [9] and [44]. One can normalize f so that it fixes the points $0, 1$, and ∞. The *normalized solution* is denoted by f^μ.

The case where μ is measurable with compact support is obtained by approximation; see [152].

To obtain a solution in the general case, suppose first that μ is measurable in $\mathbb{C}$ and that $\mu = 0$ in $\mathbb{D}$. Then

$$\mu_1(z) = \mu\left(\frac{1}{z}\right)\frac{z^2}{\bar{z}^2}$$

is measurable, $||\mu_1||_\infty = ||\mu||_\infty$ and $\mu_1 = 0$ in $\mathbb{C}\setminus\mathbb{D}$. Then (B) with μ_1 instead of μ has a normalized solution g.

Finally, given a measurable function μ in $\mathbb{C}$ with $||\mu||_\infty < 1$, we look for a mapping h such that $f = h \circ g$ where g is a homeomorphic solution of (B) with $\mu(z)\chi(z)$ instead of μ, χ being the characteristic function of $\mathbb{C}\setminus\mathbb{D}$. Then μ_h can be computed, and since $\mu_h = 0$ in $\mathbb{C}\setminus\mathbb{D}$, a homeomorphic solution of (B) with μ_h instead of μ exists. This completes the proof of existence.

3.5.3 Smoothness of the Solutions

Shabat [228] showed that for $n > 0$ and $0 < \alpha < 1$ if $\mu \in C^{n,\alpha}(D)$, i.e., the partial derivatives of order n of μ are α-Hölder continuous in D, and f is a solution of (B), then $f \in C^{n+1,\alpha}(D)$, and if $\alpha = 1$, then for all $\varepsilon > 0$, $f \in C^{n+2-\varepsilon}$.

3.5.4 Analytic Dependence on Parameters

Ahlfors and Bers [7] examined the dependence of a normalized solution of (B) on a parameter. They showed that if μ as an element of L^∞ depends analytically on a complex parameter τ, then the normalized solution of (B) in $\overline{\mathbb{C}}$ as an element of $W^{1,p}_{\text{loc}}$ depends analytically on τ (cf. [44]); see also a discussion of this subject in the monograph [51].

Chapter 4
The Degenerate Case

Recall that in our notation, a μ-homeomorphism in a domain $D, D \subset \mathbb{C}$ is an ACL homeomorphic solution of (B) in D; see Sect. 1.5. For some functions μ with $|\mu(z)| \leq 1$ a.e. and $||\mu||_\infty = 1$, there are no μ-homeomorphisms, i.e., homeomorphic solutions of (B), as illustrated below in Sects. 4.1.1 and 4.1.2. Even when a μ-homeomorphism exists, it is not known whether it is unique and generates the set of all elementary solutions. As in the classical case, by an *elementary solution*, we mean an open and discrete solution.

Contrary to the classical case where every ACL solution belongs to $W^{1,2}_{\mathrm{loc}}$, even to $W^{1,p}_{\mathrm{loc}}$ with some $p > 2$, in the relaxed classical case, an ACL solution need not be in $W^{1,2}_{\mathrm{loc}}$. Recall that the $W^{1,1}_{\mathrm{loc}}$ property implies the ACL property but not vice versa.

The main problems in the theory of μ-homeomorphisms are as in the classical case; see (3.1.1).

Approximation theorems like Theorem 2.1 and Corollary 2.3 in Chap. 2 are crucial in many existence proofs.

4.1 Examples

The following two examples show that the assumption $K \in L^p$ for some $p \geq 1$, or even that K belongs to L^p, for all $1 \leq p < \infty$, does not guarantee the existence of μ-homeomorphisms.

4.1.1 Example One

Let

$$\mu(re^{i\theta}) = -\frac{e^{2i\theta}}{1+2r}$$

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in $\mathbb{D} = \{|z| < 1\}$. Then

$$K = 1 + \frac{1}{r} \in L^1(\mathbb{D}).$$

We shall show that (B) has no μ-homeomorphic solution in $\mathbb{D}$. Indeed, consider the mapping

$$g(re^{i\theta}) = (1+r)e^{i\theta}.$$

It is easy to see that g is a μ-homeomorphism of $\mathbb{D} \setminus \{0\}$ onto $1 < |w| < 2$.

Suppose that (B) has a homeomorphic solution $f : \mathbb{D} \to \mathbb{C}$. Then, since f and g are locally quasiconformal in $\mathbb{D} \setminus \{0\}$, it follows by the classical uniqueness theorem that $h = g \circ f^{-1}$ is conformal in $f(\mathbb{D}) \setminus f(0)$ and thus can be extended to $f(\mathbb{D})$, which is impossible.

4.1.2 Example Two

Let

$$\mu(re^{i\theta}) = -e^{2i\theta} \frac{1 + \frac{1}{\log 1/r} - \frac{1}{\log^2 1/r}}{1 + \frac{1}{\log 1/r} + \frac{1}{\log^2 1/r}}.$$

in $\mathbb{D}(1/e) = \{z \in \mathbb{C} : \ |z| < 1/e\}$. Then

$$K = \log^2 \frac{1}{r} + \log \frac{1}{r} \in \bigcap_{1 \le p < \infty} L^p(\mathbb{D}(1/e)).$$

We shall show that (B) has no μ-homeomorphic solution in $\mathbb{D}(1/e)$.

Indeed, consider the mapping

$$g(re^{i\theta}) = \left(1 + \frac{1}{\log \frac{1}{r}}\right) e^{i\theta}.$$

It is easy to see that g is a μ-homeomorphism of $\mathbb{D}(1/e) \setminus \{0\}$ onto $1 < |w| < 2$. Then the assertion follows as in the previous example.

Remark 4.1. Contrary to the classical case, in the relaxed classical case as well as in the alternating case, the existence and nature of a solution may depend not only on the behavior of $|\mu|$ but also on $\arg \mu$. For instance, let μ be as in Sect. 4.1.1. Then (B) has no homeomorphic solutions in $\mathbb{D}$. However, if μ is replaced by $\tilde{\mu} = -\mu$, then (B) with $\tilde{\mu}$ instead of μ has a solution $f(re^{i\theta}) = re^{1-1/r}e^{i\theta}$ which maps $\mathbb{D}$ homeomorphically onto itself.

Furthermore, as a consequence of Proposition 3.25 in [99], Gutlyanskii, Martio, Sugawa, and Vuorinen show that given μ which is locally bounded in $\mathbb{D}\setminus\{0\}$, there is an ACL homeomorphic solution of (B) in $\mathbb{D}$ where μ is replaced by $\tilde{\mu}(z)=|\mu(z)|\bar{z}/z$.

The effect of $\arg\mu$ is noticed in [15, 264, 265] and [198] in the classical case, in [28, 148, 149, 234, 237, 238], and [99] in the relaxed classical case, and in [233, 236, 271] in the alternating case.

In the following sections, we present various conditions on $\mu(z)$ or on $K(z)$ which yield the existence of homeomorphic solutions and some of their properties.

4.2 The Singular Set

Given a measurable function μ in D with $||\mu||_\infty \le 1$, the *singular set* E of μ is the set of all points $z\in\overline{D}$ such that $||\mu|D\cap U||_\infty = 1$ for every neighborhood $U=U(z)$ of z. Note that

$$z_0\in E \;\Leftrightarrow\; \overline{\lim_{\varepsilon\to 0}}\ \operatorname*{ess\,sup}_{|z-z_0|<\varepsilon} K(z)=\infty.$$

The study of μ-homeomorphisms in the relaxed classical case can be divided into the following cases:

(1) E is specified and $E\subset\partial D$.
(2) E is specified and $E\subset D$.
(3) E is not specified.

4.3 Auxiliary Results

Throughout this chapter, for a point $a\in\mathbb{R}$ and $r>0$, $I=I(a,r)$ denotes the interval $(a-r,a+r)$, $S=S(a,r)=\{(x,y):|x-a|<r,|y|<r\}$ denotes the square centered at a with side of length $2r$, and $S^+=S^+(a,r)=S\cap\mathbb{H}$.

Let $r>0$ and ρ be measurable a.e. positive in $(0,r)$. We say that ρ is *locally bounded away from* 0 if $\rho|A\ge\rho_0(A)>0$ a.e. for all relatively compact subsets A of $(0,r)$. We say that $\rho(t)\to 0$ a.e. as $t\to 0^+$, if $\rho\to 0$ as $t\to 0^+, t\in(0,r)\setminus A$ for some set A of linear measure zero.

Let E be an open set in $\mathbb{R}$ and U a neighborhood of E in $\mathbb{C}$. We say that $|\mu(x,y)|\to 1$ *a.e. uniformly in* U as $y\to 0^+$, $x\in E$, if U has a subset A with $m_2(A)=0$ such that for all $x\in E$,

$$\lim_{y\to 0^+}|\mu(x,y)|=1,\quad (x,y)\in U\setminus A, \tag{4.3.1}$$

and there exists $C > 1$ such that for all (x_1, y) and (x_2, y) in $U \setminus A$

$$\frac{1}{C} \leq \frac{1-|\mu(x_1,y)|}{1-|\mu(x_2,y)|} \leq C. \tag{4.3.2}$$

Note that (4.3.2) is equivalent to the following condition:

$$\frac{1}{2C} \leq \frac{K(x_1,y)}{K(x_2,y)} \leq 2C. \tag{4.3.3}$$

Let μ be a locally bounded complex-valued measurable function in $\mathbb{H}$, and let g be an embedding of $\mathbb{H}$ into $\mathbb{C}$, and suppose that g is locally quasiconformal. The *deformation* $g * \mu$ of μ which is *induced by* g is defined in $g(\mathbb{H})$ by

$$g * \mu(w) = \frac{A\mu - B}{\bar{A} - \bar{B}\mu} \circ g^{-1}(w), \quad w \in g(\mathbb{H}), \tag{4.3.4}$$

where $A = \partial g$ and $B = \bar{\partial} g$ are the partial derivatives of g.

Lemma 4.1. *Let μ and g be as in Sect. 4.3. Then:*

(i) *$g * \mu$ is measurable and locally bounded in $g(\mathbb{H})$.*
(ii) *If h is a locally quasiconformal embedding of $g(\mathbb{H})$ into $\mathbb{C}$ with complex dilatation $\mu_h = g * \mu$ a.e. in $g(\mathbb{H})$, then $f = h \circ g$ is locally quasiconformal with complex dilatation $\mu_f = \mu$ a.e. in $\mathbb{H}$.*

Proof. (i) The functions $A, \bar{A}, B, \bar{B}$, and μ are measurable in $\mathbb{H}$ and hence so is $(A\mu - B)/(\bar{A} - \bar{B}\mu)$. Now, g is locally quasiconformal and thus preserves measurable sets. Consequently, $g * \mu$ is measurable in $g(\mathbb{H})$. Furthermore, since g is locally quasiconformal, and μ is locally bounded, so is $\mu_g = B/A$, and hence, by a simple estimate, it follows that $g * \mu$ is locally bounded in $g(\mathbb{H})$.

(ii) Since g and h are locally quasiconformal embeddings, so is f. Therefore, for almost all z in $\mathbb{H}$, g is differentiable at z and h at $g(z)$, and at these points an application of the chain rule yields

$$\mu_f = \frac{\bar{A}(\mu_h \circ g) - B}{\bar{B}(\mu_h \circ g) + A}. \tag{4.3.5}$$

If we now set $\mu_h = g * \mu$ and apply (4.3.5), we get $\mu_f = \mu$ a.e. in $\mathbb{H}$, as asserted. □

Lemma 4.2. *Let a be a point in $\mathbb{R}$, $r > 0$ and $\mu(x,y)$ a locally bounded measurable function in the rectangle $S^+ = S^+(a,r)$. If $|\mu(x,y)| \to 1$ a.e. uniformly in S^+ as $y \to 0^+$, $x \in (a-r, a+r)$, then there are measurable functions $\rho(y) : (0,r) \to \mathbb{R}_+$ and $M : S^+ \to \mathbb{R}$, such that $\rho(y)$ is a.e. positive and locally bounded away from 0 in $(0,r)$, and $\rho(y) \to 0$ a.e. in R_+ as $y \to 0^+$, and M satisfies*

$$\frac{1}{C} \leq M(x,y) \leq C \;\; a.e. \text{ in } \; S^+ \tag{4.3.6}$$

for some $C > 1$, *and such that for a.e.* $(x,y) \in S^+$

$$1 - |\mu(x,y)| = \rho(y)M(x,y). \tag{4.3.7}$$

Proof. Recall that by (4.3.1), there is a set A in S^+ with $m_2(A) = 0$ such that $|\mu(x,y)| \to 1$ as $y \to 0^+$, $(x,y) \in S^+ \setminus A$. For $x \in E$, let $A(x) = A \cap l(x)$, where $l(x)$ is the vertical line, which contains the point $(x,0)$. Then, by Fubini's theorem, $m_1(A(x)) = 0$ for a.e. $x \in (a-r, a+r)$.

Let x_1 be a point in $(a-r, a+r)$ such that $|\mu(x_1,y)| < 1$ for a.e. $(x_1,y) \in l(x_1)$, $|\mu(x_1,y)| \to 1$ as $y \to 0^+$ and such that $m_1(A(x_1)) = 0$. Set $\rho(y) = 1 - |\mu(x_1,y)|$. Then ρ is a measurable function in an interval $(0,r)$. Clearly, $\rho(y) \to 0$ a.e. in $(0,r)$ as $y \to 0^+$, and ρ is a.e. positive in $(0,r)$. Then ρ has a measurable extension on $\mathbb{R}$, denoted again by ρ, which is positive a.e. in $\mathbb{R}_+$. Furthermore, ρ is locally bounded in $(0,r)$, since μ is locally bounded in $\mathbb{H}$.

For $(x,y) \in S^+$, set

$$M(x,y) = \frac{1 - |\mu(x,y)|}{1 - |\mu(x_1,y)|}.$$

Then the assertion of the Lemma follows by (4.3.2). □

Lemma 4.3. *Let a be a point in $\mathbb{R}$, U a neighborhood of a in $\mathbb{C}$, and θ be a continuously differentiable function with bounded partial derivatives in $U^+ = U \cap \mathbb{H}$ such that $|\theta(z)| \le \theta_0 < \pi/2$ for all $z = x + \mathrm{i}y \in U^+$. Then:*

(i) There is a square $S = S(a,r)$, $r > 0$, $S \subset U$ such that the equation

$$\frac{\partial \xi}{\partial x}\sin\theta - \frac{\partial \xi}{\partial y}\cos\theta = 0 \tag{4.3.8}$$

has a unique solution $\xi(x,y)$ in S^+, which has a continuous extension to $S^+ \cup I$, $I = I(a-r, a+r)$, satisfying the initial condition

$$\xi(x,0) = x, \qquad x \in I. \tag{4.3.9}$$

(ii) The mapping

$$g_1(x,y) = \begin{cases} (\xi(x,y), y), & \text{for } (x,y) \in S^+ \\ (x,0), & \text{for } (x,y) \in I \end{cases} \tag{4.3.10}$$

is an embedding of $S^+ \cup I$ into $\mathbb{C}$, which is a C^1-diffeomorphism of S^+ into $\mathbb{H}$ and the identity on I.

Proof. Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -m(x,y) := -\tan\theta, \quad x + \mathrm{i}y \in U^+. \tag{4.3.11}$$

In view of the assumptions on θ, $m(x,y)$ is continuously differentiable with bounded derivatives in U^+ and hence has a continuous extension to $\overline{U^+}$, and the coefficient function $m(x,y)$ is Lipschitz in U^+ with respect to y. Therefore, there exists $\delta_0 > 0$ such that for every $\xi \in (a-\delta_0, a+\delta_0)$, (4.3.11) has a unique solution $x = \varphi(\xi, y)$, $0 \le y \le \delta_0$, which satisfies the initial condition

$$\varphi(\xi, 0) = \xi. \tag{4.3.12}$$

Since $m(x,y)$ is bounded in U^+, there exists $\delta \in (0, \delta_0)$ such that the mapping $z = G(\zeta), z = x + \mathrm{i}y, \zeta = \xi + \mathrm{i}\eta$, which is given by

$$\begin{cases} x = \varphi(\xi, \eta), \\ y = \eta, \end{cases} \tag{4.3.13}$$

is well defined in the closed rectangle

$$\overline{Q} = \{(\xi, \eta) : |\xi - a| \le \delta, 0 \le \eta \le \delta\}.$$

Clearly, $G|[a-\delta, a+\delta] = id$, and by the classical existence and uniqueness theorem for first-order ordinary differential equations, G is an embedding of $\overline{Q}$ into $\mathbb{C}$. Since $m \in C^1$, it follows that $\varphi(\xi, \eta)$ and hence also G is continuously differentiable in $Q = \mathrm{int}\overline{Q}$; see [108, Theorem 3.1.2]. Furthermore (see [108, 3.1.16]),

$$\frac{\partial x}{\partial \xi} = \exp\left(-\int_0^\eta \frac{\partial m}{\partial x}(\varphi(\xi, \eta), s)\mathrm{d}s \right) > 0,$$

and thus the Jacobian J_G of G satisfies

$$J_G = \frac{\partial(x,y)}{\partial(\xi, \eta)} = \frac{\partial x}{\partial \xi} > 0 \tag{4.3.14}$$

in Q. Therefore, G is a sense-preserving homeomorphism in $\overline{Q}$, and $G(Q) \subset \mathbb{H}$. Then there exists $r > 0$ such that $\overline{S^+}(a,r) \subset G(\overline{Q})$.

Let $g_1 = G^{-1}|\overline{S^+}(a,r)$. Then g_1 is a sense-preserving homeomorphism which is a diffeomorphism, and hence locally quasiconformal in $S^+(a,r)$, and g_1 maps $S^+(a,r)$ into Q. Furthermore, by (4.3.12),

$$g_1|(a-r, a+r) = \mathrm{i}d. \tag{4.3.15}$$

Now, (4.3.13) and the fact that $x = \varphi(\xi, y)$ is a solution of (4.3.11) imply

$$\frac{\partial \xi}{\partial x}\sin\theta - \frac{\partial \xi}{\partial y}\cos\theta = 0. \tag{4.3.16}$$

Indeed,

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} = \begin{pmatrix} x_\xi & -m \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{x_\xi} & \frac{m}{x_\xi} \\ 0 & 1 \end{pmatrix}.$$

Hence, $\xi_y/\xi_x = m = \tan\theta$, which yields (4.3.16). □

Lemma 4.4. *Let $r_1 > 0, I_1 = (a - r_1, a + r_1)$ and $S_1 = S(a, r_1)$ and let ρ be an a.e. positive measurable function in $(0, r_1)$, such that ρ is a.e. locally bounded away from 0 in $(0, r_1)$ and such that $\rho(t) \to 0$ a.e. in $(0, r_1)$ as $t \to 0^+$. Then the mapping $g_2 : S_1^+ \cup I_1 \to \mathbb{H} \cup I_1$, which is defined by*

$$g_2(\xi, \eta) = (\xi, R(\eta)), \tag{4.3.17}$$

where

$$R(\eta) = \int_0^\eta \rho(t)\mathrm{d}t \tag{4.3.18}$$

is an embedding, it is locally quasiconformal in S_1^+ and the identity on I_1.

Proof. Since ρ is measurable and a.e. positive in $(0, r_1)$, g_2 is injective and, hence, an embedding in S_1^+. Clearly, g_2 is an ACL and, hence, g_2 is a.e. differentiable in $S_1^+ \cup J$. Since ρ is locally bounded away from 0 in $(0, r_1)$, and $\mu_{g_2} = (1-\rho)/(1+\rho)$ a.e. in S_1^+, it follows that g_2 is locally quasiconformal in S_1^+. Obviously, $g_2|I_1 = \mathrm{i}d$. □

Lemma 4.5. *Let $U, U \subset \mathbb{C}$ be a neighborhood of a point $a \in \mathbb{R}$ and let $\mu(x,y)$ be a complex-valued measurable function in $U^+ = U \cap \mathbb{H}$. If μ is locally bounded in U^+ and:*

(a) $|\mu(x,y)| \to 1$ *a.e. uniformly in U^+ as $y \to 0^+$, $x \in U \cap \mathbb{R}$,*
(b) $\arg\mu$ *is continuously differentiable with bounded partial derivatives in U^+,*
(c) $|\arg\mu| \le 2\theta_0 < \pi$ *in U^+,*

then there are a square $S_0 = S(a, r_0)$ and an embedding g_0 of $S_0^+ \cup I_0$, where $I_0 = (a - r_0, a + r_0)$, into $\mathbb{H}$, which is locally quasiconformal in S_0^+ and is the identity on I_0 such that the deformation $\mu_2 = g_0 * \mu$ of μ which is induced by g_0 is bounded in $g_0(S_0^+)$.

Proof. In view of condition (a) of the lemma, by Lemma 4.2, there are measurable functions $\rho : (0, r) \to \mathbb{R}_+$ and $M : S^+ \to \mathbb{R}$ such that for a.e. $(x,y) \in S^+$,

$$\mu(x,y) = \mathrm{e}^{\mathrm{i}\arg\mu(x,y)}[1 - \rho(y) \cdot M(x,y)], \tag{4.3.19}$$

and such that (4.3.6) and (4.3.7) hold.

Set $\theta(z) = \arg\mu(z)/2$. Then θ satisfies the assumptions of Lemma 4.3. Let $r > 0$, $S = S(a,r)$, $I = I(a,r)$, and g_1 be as in Lemma 4.3. Then g_1 is the identity on I and maps S^+ homeomorphically into $\mathbb{H}$. Next, choose $r' > 0$ such that $S'^+ \subset$

$g_1(S^+)$, where $S' = S(a,r')$ and let $I' = S' \cap \mathbb{R}$. Choose $r_1 \in (0,r')$ such that for $S_1 = S(a,r_1), S_1^+ \subset g_1(S')$.

Now, let $A_1 = \partial g_1/\partial z, B_1 = \partial g_1/\partial \bar{z}$ and

$$C_1 = A_1 e^{i\theta} - B_1 e^{-i\theta}. \tag{4.3.20}$$

Then

$$\begin{aligned} A_1 &= \frac{1}{2}[\xi_x + 1 - i\xi_y],\ B_1 = \frac{1}{2}[\xi_x - 1 + i\xi_y], \\ \Im\, C_1 &= \xi_x \sin\theta - \xi_y \cos\theta, \end{aligned} \tag{4.3.21}$$

and thus, in view of (4.3.8), C_1 is real.

By (4.3.20), $|C_1| \geq |A_1| - |B_1| > 0$, and since $|A_1|^2 - |B_1|^2 = J_{g_1} > 0$, it follows that $C_1 \neq 0$, and thus, by straightforward computations,

$$e^{i\theta}\left(\frac{A_1}{C_1} + \frac{\bar{B}_1}{\bar{C}_1}\right) = \frac{J_{g_1}}{C_1\bar{C}_1} = J_{g_1}|C_1|^{-2} > 0. \tag{4.3.22}$$

Let $\mu_1 = g_1 * \mu$ be the deformation of the dilatation μ, which is induced by g_1. By setting in (4.3.4) the expression for μ which is given in (4.3.19), one obtains that a.e. in S_1^+

$$\mu_1 = \frac{e^{2i\theta}(1-\rho M)A_1 - B_1}{\bar{A_1} - e^{-2i\theta}(1-\rho M)\bar{B_1}} \circ G_1 = \frac{C_1 - e^{i\theta}\rho M A_1}{\bar{C_1} + e^{i\theta}\rho M \bar{B_1}} \circ G_1, \tag{4.3.23}$$

where $G_1 = g_1^{-1}$. Since $C_1 \neq 0$ and real and $\rho(\eta) \to 0$ a.e. as $\eta \to 0$,

$$\mu_1 = \left[\frac{C_1}{\bar{C_1}}\left(1 - e^{i\theta}\left(\frac{A_1}{C_1} + \frac{\bar{B_1}}{\bar{C_1}}\right)M\rho + \rho\cdot o(\rho)\right)\right] \circ G_1 \tag{4.3.24}$$

holds a.e. for a smaller rectangle, which we denote again by S_1^+, $S_1 = S(a,r_1)$. Thus,

$$\mu_1(\zeta) = 1 - \rho(\eta)M_1(\zeta) \quad \text{a.e. in } S_1, \tag{4.3.25}$$

where, in view of (4.3.22),

$$M_1 = MJ_{g_1}|C_1|^{-2} + o(\rho)$$

is a real-valued measurable function, which satisfies

$$\frac{1}{R_1} \leq |M_1| \leq R_1 \tag{4.3.26}$$

for some $R_1 > 0$.

Next, let g_2 be as in Lemma 4.4 and let $\mu_2 = g_2 * \mu_1$ be the deformation of μ_1, which is induced by $g_2|S_1^+$. Since μ_1 is locally bounded in S_1^+ and g_2 is locally quasiconformal, it follows, by Lemma 4.1, that μ_2 is measurable and locally bounded in $g_2(S_1^+)$.

We now compute and estimate μ_2. Let $A_2 = (g_2)_\zeta$ and $B_2 = (g_2)_{\bar\zeta}$. Then

$$A_2 = \frac{1}{2} + \frac{\rho(\eta)}{2} \quad \text{and} \quad B_2 = \frac{1}{2} - \frac{\rho(\eta)}{2}$$

a.e. in S_1^+. To obtain μ_2, we plug $\mu_1 = 1 - \rho(\eta)M_1(\zeta)$ of (4.3.25) in (4.3.4) and get

$$\mu_2 \circ g_2 = \frac{(1+\rho)(1-\rho M_1) - 1 + \rho}{1+\rho-(1-\rho)(1-\rho M_1)} = \frac{2-M_1}{2+M_1} + o(\rho) \quad \text{a.e. in } S_1^+$$

as $\rho \to 0$. In view of (4.3.26) and the fact that $\rho(\eta) \to 0$ a.e. as $\eta \to 0^+$, there is $r_2 \in (0, r_1)$ such that for $S_2 = S(a, r_2), S_2^+ \subset g_2(S_1^+)$, and such that

$$\|\mu_2\|_\infty < 1 \ \text{ in } \ S_2^+. \tag{4.3.27}$$

Let $S_0 = S(a, r_0)$ and $I_0 = S_0 \cap \mathbb{H}$. Choose $r_0 > 0$ such that $S_0^+ \subset g_1^{-1}(S_2^+)$.

For $(x,y) \in S_0 \cup I_0$, set $g_0 = g_2 \circ g_1$. Then g_0 is an embedding of $S_0^+ \cup I_0$ into $\mathbb{H}$. Since g_1 and g_2 are locally quasiconformal in S_0^+ and S_2^+, respectively, and $g_1(S_0^+) \subset S_2^+$, it follows that g_0 is locally quasiconformal in S_0^+. Since $g_1|I_0 = \mathrm{i}d = g_2|S_2 \cap \mathbb{R} = id$, it follows that $g_0|I_0 = \mathrm{i}d$.

Finally, by Lemma 4.1 (ii), $g_0 * \mu = (g_2 \circ g_1) * \mu = g_2 * (g_1 * \mu) = g_2 * \mu_1 = \mu_2$. Then the assertion follows by (4.3.27). □

4.4 Case (i): The Singular Set E is Specified and $E \subset \partial D$

4.4.1 Existence and Uniqueness

Note that $E \subset \partial D$ means that μ is locally bounded in D. Bers [33], 3.1 (see also [44] and [30]), showed that in this case a μ-homeomorphism exists and is unique up to a postcomposition by a conformal mapping. The proof follows by a normal families argument or by application of the uniformization theorem.

4.4.2 Boundary Behavior

We consider the case that $D = \mathbb{H} := \{z \in \mathbb{C} : \Im\, z > 0\}$ and $E \subset I\!R \subset \partial\mathbb{H} = I\!R \cup \{\infty\}$. Let f be a μ-homeomorphism of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im\, z > 0\}$ into $\mathbb{C}$. Note that f can be viewed as a conformal embedding of $\mathbb{H}$ endowed with the measurable conformal structure $\mathrm{d}s = |\mathrm{d}z + \mu(z)\mathrm{d}\bar{z}|$ into $\mathbb{C}$. It is well known that if f is conformal in $\mathbb{H}$ (endowed with the Euclidean metric), then $f(\mathbb{H})$ is a proper subset of $\mathbb{C}$, and if, in addition, $f(\mathbb{H})$ is a Jordan domain, f has a homeomorphic extension to $\overline{\mathbb{H}}$, the closure of $\mathbb{H}$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The same is true for a μ-homeomorphism in $\mathbb{H}$ when $||\mu||_\infty < 1$.

If μ and f are as above, $f(\mathbb{H})$ is either $\mathbb{C}$ (see Example 2 below) or a proper subset of $\mathbb{C}$. Thus, one may ask under what further conditions on μ every μ-homeomorphism f satisfies:

(1) $f(\mathbb{H}) \neq \mathbb{C}$.
(2) If, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension to $\overline{\mathbb{H}}$.

If μ is measurable and *locally bounded in* $\mathbb{H}$, i.e., $||\mu|A||_\infty < 1$ for every relatively compact subset A in $\mathbb{H}$, a μ-homeomorphism of $\mathbb{H}$ exists, and its image may be either $\mathbb{C}$ (see 4.2 below) or a proper subset of $\mathbb{C}$. One may ask under what conditions on μ, with or without the assumption that μ is locally bounded, every μ-homeomorphism f satisfies:

(1) $f(\mathbb{H}) \neq \mathbb{C}$.
(2) If, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\overline{\mathbb{H}}$ = the closure of $\mathbb{H}$ in $\overline{\mathbb{C}}$.

The main results in this section are the following Theorems 4.1 and 4.2, which were established in [233].

Theorem 4.1. *Let f be an embedding of $\mathbb{H}$ into $\mathbb{C}$ and $a \in \mathbb{R}$. Suppose that a has a neighborhood U in $\mathbb{C}$ such that f is* ACL *in $U^+ = U \cap \mathbb{H}$ with a locally bounded complex dilatation $\mu = \mu_f$ in U^+. If:*

(a) $|\mu(x,y)| \to 1$ *a.e. uniformly in U^+ as $y \to 0^+$, $x \in U \cap \mathbb{R}$,*
(b) $\arg \mu$ *is continuously differentiable with bounded partial derivatives in U^+,*
(c) $|\arg \mu| \leq 2\theta_0 < \pi$ *in U^+,*

then

(i) $f(\mathbb{H}) \neq \mathbb{C}$.
(ii) If, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension to $\mathbb{H} \cup (U \cap \mathbb{R})$.

Theorem 4.2. *Let E be an open set in $\mathbb{R}$, possibly $E = \mathbb{R}$, $\mu : \mathbb{H} \to \mathbb{C}$ a locally bounded measurable function, and $f : \mathbb{H} \to \mathbb{C}$ a μ-homeomorphism. Suppose that every point $a \in E$ has a neighborhood $U, U \subset \mathbb{C}$ such that conditions (a)–(c) of Theorem 4.1 hold with θ_0 depending on U.*

If $f(\mathbb{H})$ is a Jordan domain, and if f is quasiconformal in every domain $D, D \subset \mathbb{H}$ whose closure in $\overline{\mathbb{C}}$ is contained in $\overline{\mathbb{H}} \setminus E$, then f has a homeomorphic extension to $\overline{\mathbb{H}}$.

Suppose that μ satisfies the assumptions of Theorem 4.2 with $E = \mathbb{R}$ and that f is a μ-homeomorphism of $\mathbb{H}$ onto a Jordan domain. Then, by Theorem 4.2 and in view of the fact that $f(\mathbb{H})$ is a Jordan domain in $\mathbb{C}$, it follows that f has a homeomorphic extension to $\mathbb{H} \cup \mathbb{R}$; however, f need not have a homeomorphic extension to $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ as illustrated in (4.4.5).

4.4.3 Proof of Theorem 4.1

Let a, f, and μ be as in the theorem. Then, by Lemma 4.4, there is an embedding g_0 of $S_0^+ \cup I_0$ which is qc in S_0^+ and the identity on I_0 and such that the deformation $\mu_2 = g_0 * \mu$ is bounded in $g_0(S_0^+)$. Hence, a quasiconformal mapping $\varphi : g_0(S_0^+) \to \mathbb{C}$ with $\mu_\varphi = \mu_2$ a.e. in $g_0(S_0^+)$ exists.

By Lemma 4.1, for a.e. $z \in S_0^+$,

$$\mu_f(z) = \mu(z) = \mu_{\varphi \circ g_0}.$$

Hence, there exists a conformal mapping h of $(\varphi \circ g_0)(S_0^+) = \varphi(g_0(S_0^+))$ onto $f(S_0^+)$.

The cluster set of f on I_0 is the same as the cluster set of $h \circ \varphi$ on I_0. The latter one meets $\mathbb{C}$ since $h \circ \varphi$ is quasiconformal. It thus follows that the cluster set of f on $\mathbb{R}$ meets $\mathbb{C}$, and therefore, $f(\mathbb{H}) \neq \mathbb{C}$, and (i) follows.

Suppose now that $f(\mathbb{H})$ is a Jordan domain. Then $f(S_0^+)$ is a Jordan domain, and since $h \circ \varphi$ is quasiconformal, it has a homeomorphic extension on $S_0^+ \cup I_0$. Now $g_0|I_0 = id$ hence, $f = (h \circ \varphi) \circ g_0$ has a homeomorphic extension on I_0.

The same argument can be applied to any other point in $U \cap \mathbb{R}$, from which we can conclude that if $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension to $\mathbb{H} \cup (U \cap \mathbb{R})$, and thus (ii) follows.

4.4.4 Proof of Theorem 4.2

Let E, μ, and f be as in the theorem. If $E = \emptyset$, then f is quasiconformal in $\mathbb{H}$, and the assertion follows by the classical qc theory.

Suppose that $E \neq \emptyset$. Then E is a countable union of disjoint open intervals I. Fix one of these intervals, say I. Then, by Theorem 4.1, every point in I has an interval, which is contained in I where f extends homeomorphically, and since $f(\mathbb{H})$ is a Jordan domain, f has a homeomorphic extension to $\mathbb{H} \cup I$ and consequently to $\mathbb{H} \cup E$.

We now show that f has a continuous extension to $\mathbb{H} \setminus E$. Let b be a point in the $(\mathbb{R} \cup \{\infty\}) \setminus E$. Since $\overline{\mathbb{C}}$, and hence $\overline{f(\mathbb{H})}$, is compact, the cluster set of f at b is not nonempty. It suffices to show that it is nondegenerate. Suppose that it is not nondegenerate. Then there are sequences x_n and x'_n in $\mathbb{H}$ which tend to b such that $f(x_n)$ and $f(x'_n)$ tend to different limits, say w and w', respectively. Let K and K' be disjoint arcs in $\overline{f(\mathbb{H})}$, the first one containing all points $f(x_n)$ and the other one all $f(x_n')$, and such that each of these two arcs lies in $f(\mathbb{H})$ except for its end point, which is w and w', respectively. Now choose a domain D with $\overline{D} \subset \mathbb{H} \cup \{b\}$, which contains $f^{-1}(K) \cup f^{-1}(K')$, and such that the family Γ of all paths in D which join $f^{-1}(K)$ and $f^{-1}(K')$ has infinite modulus $M(\Gamma) = \infty$. The path family $f(\Gamma)$ is a subfamily of all paths joining K and K' in $\overline{\mathbb{C}}$, and the latter one has finite modulus, since K and K' are disjoint continua. Therefore, the modulus $M(f(\Gamma))$ of $f(\Gamma)$ is finite, which is impossible since f is quasiconformal in D and $M(\Gamma) = \infty$.

Suppose now that J is a nondegenerate component of $\mathbb{R} \cup \{\infty\} \setminus E$. Choose a domain $D, D \subset \mathbb{H}$ such that $J \subset \partial D \cap \mathbb{R}$. Since f is quasiconformal in D, and $f(J)$ is a free boundary arc in $f(D)$, it follows that f has a homeomorphic extension to the union of $\mathbb{H}$ and the interior (in the R topology) of J and thus (in view of the existence of a continuous extension at each boundary point) on $\mathbb{H} \cup J$.

Recalling that $f(D)$ is a Jordan domain, we obtain that f has a homeomorphic extension to $\overline{\mathbb{H}}$.

4.4.5 *Examples*

4.4.5.1 Example 1

In the following example, f and μ satisfy the conditions of Theorem 4.1 with $E = \mathbb{R}$. Here, f maps $\mathbb{H}$ homeomorphically onto itself, and it has a homeomorphic extension onto $\mathbb{H} \cup \mathbb{R}$, as asserted in the theorem, but not onto $\overline{\mathbb{H}}$. More precisely, here $f(\mathbb{H} \cup \mathbb{R}) = \mathbb{H} \cup \mathbb{R}_+ \cup \{\infty\}$, and the cluster set of f at ∞ is the closed ray $[-\infty, 0]$.

The mapping f in this example is defined by

$$f(x,y) = \mathrm{e}^{x + 2\mathrm{i} \arctan y^2}, \qquad y > 0.$$

Then the complex dilatation of f is given by

$$\mu(x,y) = \frac{y^2 - 4y + 1}{y^2 + 4y + 1}$$

and

$$1 - |\mu| = 1 - \mu = \frac{4y}{y^2 + 4y + 1}.$$

Therefore, $1 - |\mu(x,y)| = \rho(y) \cdot M(x,y)$, where

$$\rho(y) = 4y \qquad \text{and} \qquad M(x,y) = \frac{1}{y^2 + 4y + 1}.$$

Then condition (a) holds at every point $x \in \mathbb{R}$ with $C = 1$ in (4.3.2). Also $\arg \mu = 0$, and thus conditions (b) and (c) hold too.

4.4.5.2 Example 2

In the following example, f is a μ-homeomorphism in $\mathbb{H}$ with μ satisfying conditions (a) and (b) in Theorem 4.1, but not condition (c). In this example, f maps $\mathbb{H}$ onto $\mathbb{C}$.

The mapping f in this example is defined by

$$f(x,y) = x + \mathrm{i}\log\, y, \qquad y > 0.$$

Then the complex dilatation of f is given by

$$\mu(x,y) = \frac{y-1}{y+1},$$

and

$$1 - |\mu| = \frac{2y}{y+1} = \rho(y) \cdot M(x,y)$$

where

$$\rho(y) = 2y, \quad M(x,y) = \frac{1}{y+1}, \quad \text{and} \quad \arg\mu(x,y) \to \pi \text{ as } y \to 0^+.$$

Thus (a) and (b) hold, and (c) fails.

4.4.5.3 Example 3

In the following example, f is a μ-homeomorphism in $\mathbb{H}$ with a complex dilatation μ satisfying conditions (a) and (b) in Theorem 4.1, but not condition (c). Here, f maps $\mathbb{H}$ onto itself, but f has no injective and hence no homeomorphic extension to $\mathbb{H} \cup \mathbb{R}$.

The mapping f in this example is defined by

$$f(x,y) = xy + \mathrm{i}y, \qquad y > 0.$$

Then

$$\mu(x,y) = \frac{-1+y+\mathrm{i}x}{1+y-\mathrm{i}x},$$

$$1 - |\mu| = \frac{1-|\mu|^2}{1+|\mu|} = \frac{4y}{(y+1)^2+x^2} \cdot \frac{1}{1+|\mu|} = \rho(y) \cdot M(x,y)$$

where

$$\rho(y) = 4y \quad \text{and} \quad M(x,y) = \frac{4y}{(y+1)^2+x^2} \cdot \frac{1}{1+|\mu|}.$$

Note that μ satisfies conditions (a) and (b) in Theorem 4.1, and $\mu(x,y) \to -1$ as $y \to 0^+$, and thus condition (c) fails. Here, for every $x \in \mathbb{R}$, $f(x,y) \to 0$ as $y \to 0^+$.

4.4.5.4 Example 4

In the following example, f is a μ-homeomorphism in $\mathbb{H}$ with μ satisfying conditions (b) and (c) in Theorem 4.1, but not condition (a). Here, f maps $\mathbb{H}$ onto itself, but f has no limit at $x = 0$ and hence has no homeomorphic extension to $\mathbb{H} \cup \mathbb{R}$.

The mapping f in this example is defined as follows: For $y > 0$, let $f(x,y) = u(x,y) + \mathrm{i}y^2$, where for $x \geq 0$

$$u(x,y) = \begin{cases} x/y & \text{if } x \leq y/2, \\ x/y + \frac{y-1}{2}(x/y - 1/2)^2 & \text{if } y/2 \leq x \leq 3y/2, \\ 1 + x - y & \text{if } x > 3y/2, \end{cases}$$

and for $y > 0$ and $x < 0$

$$u(x,y) = -u(-x,y).$$

By checking the values of $u(x,y)$ and its partial derivatives at each of the five sectors which are defined by $|x| < y/2$, $y/2 < |x| < 3y/2$ and $|x| > 3y/2$ and their limits at points on the lines $|x| = y/2$ and $|x| = 3y/2$, $y > 0$, it is not hard to verify that f is a C^1-homeomorphism in $\mathbb{H}$.

One can compute $\arg\mu(z)$ and $\partial \arg\mu(z)/\partial y$ in $\mathbb{H}$ (say with aid of a tool like Maple) and verify that there is $r > 0$ such that conditions (b) and (c) hold in each of the rectangles $S^+(x,r), x \in \mathbb{R}$.

Obviously, for $x \in \mathbb{R}$, $|\mu(x,y)| \to 1$ as $y \to 0^+$. Simple computations show that near every point $x > 0$ in $\mathbb{R}$, and therefore near every point $x < 0$ in $\mathbb{R}$,

$$1 - |\mu(x,y)|^2 = \frac{8y}{(1+2y)^2 + 1}.$$

Hence, near each of these points

$$1 - |\mu(x,y)| = O(y) \quad \text{as } y \to 0^+.$$

By similar computations, one obtains that for points (x,y) in the middle sector $|x| < y/2$

$$1 - |\mu(x,y)|^2 = \frac{8y^4}{x^2 + y^2(1+2y^2)^2}$$

and thus,

$$1 - |\mu(x,y)| = o(y) \quad \text{as } y \to 0^+$$

in the middle sector. Therefore, condition (a) fails at the point 0. Note that condition (a) holds near every other point in $\mathbb{R}$.

As noted above, f maps $\mathbb{H}$ onto itself. Clearly, the cluster set of f at 0 is the line segment $|x| \leq 1, y = 0$. Hence, f has no homeomorphic extension to $\mathbb{H} \cup \mathbb{R}$.

4.5 Case (ii): The Singular Set *E* is Specified and $E \subset D$

Srebro and Yakubov [234] proved the following existence and uniqueness theorem, in the case when $D = \mathbb{C}$ and $E = \mathbb{R}$:

Theorem 4.3. *Let μ be a complex valued measurable function such that μ is locally bounded in $\mathbb{C} \backslash \mathbb{R}$. Suppose that for every point $x_0 \in \mathbb{R}$ there are positive numbers $r = r(x_0)$ and $R = R(x_0)$ such that for a.e. $z = x + \mathrm{i}y$ in the disk $D(x, r_0) = \{z : |z - x_0| < r\}$*

$$\mu(z) = \mathrm{e}^{2\mathrm{i}\theta(z)}[1 - \rho(y)M(z)], \tag{4.5.1}$$

for some continuously differentiable function $\theta : D(x_0, r) \to (-\pi/2, \pi/2)$, continuous function $\rho : (-r, r) \to \mathbb{R}$, with $\rho(0) = 0$ and $\rho(y) > 0$ for all $y \neq 0$, and measurable function $M : D(x_0, r) \to \mathbb{C}$ such that the following condition (R) holds a.e. in $D(x_0, r)$:

$$\frac{1}{R} \leq \Re\, M(z) \leq R \ \text{ and } |\Im\, M(z)| < R. \tag{4.5.2}$$

Then

(i) A μ-homeomorphism $f : \mathbb{C} \to \mathbb{C}$ exists.
(ii) f is unique up to a postcomposition of f by a conformal map.
(iii) f generates the set of all $W^{1,2}_{\mathrm{loc}}$ solutions.

In [238], the existence and uniqueness of a μ-homeomorphism are proved in the case where E is a union of a finite number of smooth closed arcs or crosscuts in a domain D under the following assumptions on D and E.

Let D be a domain in $\mathbb{C}$, and let E be a relatively closed subset of D. Suppose that each point z_0 in E has a neighborhood $U(z_0)$ such that $E \cap U(z_0)$ is either a C^3 arc or a *finitely star* with a vertex at z_0 and end points in $\partial U(z_0)$, meaning that $E \cap U(z_0)$ is a finite union of arcs, each connecting the point z_0 with $\partial U(z_0)$ and which are disjoint except for their common end point z_0.

In the latter case, we assume also that each of the arms is C^3 and that the arms meet at non-zero angles.

According to the assumptions, E is a countable union of arcs E_n with end points in the union of the boundary of D in $\overline{\mathbb{C}}$ and the set of all star vertices of E. Furthermore, since the arms are C^3, each E_n excluding its end points has a neighborhood $U_n, U_n \subset D$, such that the U_n's are disjoint, and for every inner point z_0 in E_n, there are $r = r(z_0) > 0$ and a C^3 path $\gamma : [-r, r] \to E_n$ with $\gamma(0) = z_0$ and $|\gamma'(s)| = 1, |s| \leq r$ such that the normals at points of γ do not intersect in U_n.

Theorem 4.4. *Let D, E, E_n and U_n be as 3.19. Let $\mu(z)$ be a complex valued measurable function in D which is locally bounded in $D \setminus E$.*

Suppose that for each $n = 1, 2, \ldots$, there are:

(a) A C^1 function $\theta_n : U_n \to \mathbb{R}$
(b) A continuous function $\rho_n : \mathbb{R} \to [0, \infty)$ with $\rho_n(0) = 0$
(c) A measurable function $M_n : U_n \to \mathbb{C}$
(d) A positive number R_n such that for a.e. $z \in U_n$

$$1/R_n \leq \Re\, M_n(z) \leq R_n \quad \text{and} \quad |\Im\, M_n(z)| \leq R_n, \tag{4.5.3}$$

$$\mu(z) = \mathrm{e}^{2\mathrm{i}\theta_n(z)}[1 - \rho_n(\mathrm{dist}(z, E_n))M_n(z)] \tag{4.5.4}$$

and such that

$$\mathrm{d}z + \mu(z)\mathrm{d}\bar{z} = \mathrm{d}z + \mathrm{e}^{2\mathrm{i}\theta(z)}\mathrm{d}\bar{z} \neq 0$$

when restricted to E_n. Then

(i) A μ-homeomorphism $f : D \to \mathbb{C}$ exists.
(ii) f is unique up to a postcomposition of f by a conformal map.
(iii) f generates the set of all $W^{1,2}_{\mathrm{loc}}$ solutions.

Theorem 4.5. *Let D, E, E_n, U_n be as in Theorem 4.4. Let $U_{n,1}$ and $U_{n,2}$ be the connected components of $U_n \setminus E_n$, and let $U = \bigcup U_n$. Let $\mu(z)$ be a complex-valued measurable function in D which is locally bounded in $D \setminus E$. Suppose that μ is as in Theorem 4.4, except for some of the sets U_n where either $||\mu|U_{n,1}||_\infty < 1$ or $||\mu|U_{n,2}||_\infty < 1$. Then the assertions (i)–(iii) of Theorem 4.4 hold.*

The main tool in proving these existence and uniqueness theorems is the method of the deformation of the complex dilatation.

4.6 Case (iii): The Singular Set *E* Is Not Specified

In this case, the control on μ which gives existence and uniqueness and allows a study of properties may be expressed either by integral estimates like the convergence or divergence of

$$\int_U F(\mu)\,\mathrm{d}m \quad \text{or} \quad \int_U \Phi(K_\mu)\,\mathrm{d}m, \quad U \subset D$$

for some functions F or Φ or by measure estimates like

$$|\{z \in D : |\mu(z)| > 1 - \varepsilon\}| \leq \varphi(\varepsilon) \quad \text{or} \quad |\{z \in D : K_\mu(z) > t\}| \leq \psi(t)$$

for some functions φ or ψ. These two basic forms of control are related. Indeed, suppose that

$$\Phi : [1,\infty) \to \mathbb{R}_+$$

is a nondecreasing function such that for a Borel $B \subset D$,

$$\int_B \Phi(K(z))\,\mathrm{d}m \le M_B < \infty.$$

For $t > 1$, set

$$E = E_t = \{z \in B : K(z) > t\}.$$

Then

$$\begin{aligned}\Phi(t)|E| &\le \int_E \Phi(K(z))\,\mathrm{d}m \\ &= \int_E \Phi(K(z))\,\mathrm{d}m + \int_{B\setminus E} \Phi(K(z))\,\mathrm{d}m = \int_B \Phi(K(z))\,\mathrm{d}m \le M_B.\end{aligned}$$

Hence,

$$|E_t| \le \frac{M_B}{\Phi(t)} := \Psi(t).$$

We first survey results involving integral estimates and then results involving measure estimates.

4.6.1 Pesin

Pesin [185] established the existence of a μ-homeomorphism f in the unit disc $\mathbb{D}$ when

$$\int_{\mathbb{D}} \mathrm{e}^{[K(z)]^p}\mathrm{d}m(z) < \infty$$

for some $p > 1$. He also proved that under this condition,

$$f \in \bigcup_{q<2} W^{1,q}_{\mathrm{loc}}(\mathbb{D}) \qquad \text{and} \qquad f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{D})\,.$$

4.6.2 Miklyukov and Suvorov

Miklyukov and Suvorov [174] obtained the existence of a μ-homeomorphism $f : \mathbb{D} \to \mathbb{C}$ when

$$K(z) \le K_0 + Q(z)$$

for some positive constant K_0 and a function $Q(z) \in W_0^{1,2}$. They also showed that f is unique and $f^{-1} \in W_{\mathrm{loc}}^{1,2}(f(\mathbb{D}))$. Recently, Martio and Miklyukov [158] generalized this result to the case where $K(z)$ is dominated by some function $Q(z)$ which belongs to $W_{\mathrm{loc}}^{1,2}(\mathbb{D})$; see Sects. A.3 and A.4 below.

4.6.3 Lehto

Lehto [148] and [149] considered the case, where μ is locally bounded in $\mathbb{C} \backslash E$ for some compact set E in D with $m_2(E) = 0$, and defined for $z \in \mathbb{C}$ and $r > 0$

$$\psi_\mu(z,r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|1 - \mathrm{e}^{-2\mathrm{i}\theta} \mu(z + r\mathrm{e}^{\mathrm{i}\theta})\right|^2}{1 - |\mu(z + r\mathrm{e}^{\mathrm{i}\theta})|^2} \mathrm{d}\theta.$$

By using results of Andreian Cazacu [15] and Reich and Walczak [198], Lehto showed in [148] that if for all z in D and $0 < r_1 < r_2$,

$$I(r_1, r_2, r) = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r\psi_\mu(z,r)} \to \infty \qquad \text{as } r_1 \to 0 \text{ or as } r_2 \to \infty, \tag{4.6.1}$$

then there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that for every relatively compact domain U in $\mathbb{C} \setminus E$, $f|U$ is a solution of (B).

Cima and Derrick [67] showed that (4.6.1) is not a necessary condition for existence.

4.6.4 Brakalova and Jenkins

Brakalova and Jenkins [56] modified Pesin's condition and proved that given a measurable function μ in $\mathbb{C}$ with $||\mu||_\infty \leq 1$, a μ-homeomorphism f of $\mathbb{C}$ onto itself with $f(0) = 0$ and $f(1) = 1$ exists if the following two conditions hold:

$$\iint_B \exp \frac{\frac{1}{1-|\mu|}}{1 + \log\left(\frac{1}{1-|\mu|}\right)} \, \mathrm{d}A < \Phi_B \tag{4.6.2}$$

for every bounded measurable set B in D, where Φ_B is a constant which depends on B, and

$$\iint_{\{|z|<R\} \cap D} \frac{1}{1 - |\mu|} \, \mathrm{d}A = O(R^2), \quad R \to \infty. \tag{4.6.3}$$

They also showed that $f \in \bigcup_{1\le p<2} W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ and that f generates the set of all $W^{1,2}_{\mathrm{loc}}$ homeomorphic solutions. The proof is based on estimates by the Reich and Walczak result on the change of the modulus of a ring under a quasiconformal mapping and on an approximation theorem; see Sect. A.1 below.

4.6.5 Iwaniec and Martin

In [116], Iwaniec and Martin studied wider classes of μ's with $K(z)$ satisfying exponentially and subexponentially integrability conditions. They introduced the term "principal solution." A homeomorphism $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called a *principal solution* if there is a discrete set E such that $h \in W^{1,1}_{\mathrm{loc}}(\mathbb{C}\setminus E)$, h satisfies (B) a.e., and at ∞, h has the normalization

$$h(z) = z + o(1).$$

Theorem 4.6. [116]. *There exists a number $p_0 > 1$ such that if*

$$|\mu(z)| \le \frac{K(z)-1}{K(z)+1}\chi_{\mathbb{D}}$$

and

$$\mathrm{e}^K \in L^p(\mathbb{D})$$

for $p \ge p_0$, then (B) has a unique principal solution $h \in z + W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

Theorem 4.7. [116]. *There exists a number $p_* \ge 1$ such that if*

$$\frac{K(z)}{1+\log K(z)} \in L^p(\mathbb{D})$$

for $p \ge p_*$, then (B) has a unique principal solution $h \in z + W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$ with Orlicz function $Q(t) = t^2 \log^{-1}(e+t)$.

The proofs of these results are based on a monotonicity principle in the Orlicz–Sobolev classes and deep results in harmonic analysis.

For uniqueness of a principal solution in $z + W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$, Iwaniec and Martin [116] looked at $f = h_1 - h_2$, where h_1 and h_2 are assumed to be principal solutions of (B) in $W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$. Then f is of bounded distortion, satisfies (B), and its differential Df is in L^p. By a Liouville theorem, which they establish for this class of mappings, $f = const$, and in view of the normalization of h_1 and h_2, $f = 0$.

4.6.6 Gutlyanskii, Martio, Sugawa, and Vuorinen

By using [198], Gutlyanskii, Martio, Sugawa, and Vuorinen [99] provided conditions for the existence of a $W^{1,1}_{\text{loc}}$ μ-homeomorphism where both $|\mu|$ and $\arg\mu$ are involved; see Sect. A.2 below.

They also studied the properties of μ-homeomorphisms of $\overline{\mathbb{C}}$ in terms of what they call the angular dilatation D_{μ,z_0} and a dominating factor, which are defined as follows:

Given $z_0 \in D$, the *angular dilatation* of μ is defined by

$$D_{\mu,z_0} = \frac{|1-\mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}|^2}{1-|\mu(z)|^2} \text{ if } z\in\mathbb{C}, \quad \text{and} \quad D_{\mu,z_0}(z) = D_{\mu,0}(z) \text{ if } z_0=\infty.$$

A function $H : [0,+\infty) \to \mathbb{R}$ is called a *dominating factor* if the following conditions are satisfied:

(1) $H(x)$ is continuous and strictly increasing in $[x_0,+\infty)$ and $H(x) = H(x_0)$, for $x \in [0,x_0]$ for some $x_0 \geq 0$.
(2) The function $\mathrm{e}^{H(x)}$ is convex in $x \in [0,+\infty)$.

A dominating factor H is said to be of *divergence type* if

$$\int_1^\infty \frac{H(x)}{x^2}\mathrm{d}x = +\infty.$$

Theorem 4.8. [99]. *Suppose that μ is a measurable function in $\mathbb{C}$ with $|\mu(z)| < 1$ a.e. such that $K \in L^p_{\text{loc}}(\mathbb{C})$ for some $p > 1$ and for each point $z_0 \in \mathbb{C}$ either $D_{\mu,z_0} \leq M$ a.e. in the disc $B(z,r_0) = \{|z-z_0| < r_0\}$ or*

$$\int_{B(z_0,r_0)} \mathrm{e}^{H(D_{\mu,z_0}(z))}\, \mathrm{d}m(z) \leq M, \quad \text{if } z_0 \in \mathbb{C}$$

and

$$\int_{|z|>r_0} \mathrm{e}^{H(D_{\mu,z_0}(z))} \frac{\mathrm{d}m(z)}{|z|^4} \leq M, \quad \text{if } z_0 = \infty$$

for some dominating factor $H = H_{z_0}$ of divergence type and positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$.

Then there exists a normalized homeomorphic solution $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of (B) such that $f \in W^{1,q}_{\text{loc}}(\mathbb{C}), q = 2p/(1+p)$, and $f^{-1} \in W^{1,2}_{\text{loc}}$.

The following theorem may be viewed as an extended version of the theorems of [56] and [116]:

Theorem 4.9. [99]. *Let H be a dominating factor of divergence type. Suppose that μ is a measurable function in $\mathbb{C}$ with $||\mu||_\infty \le 1$ which satisfies*

$$\int_{\mathbb{C}} e^{H(K(z))} d\sigma < +\infty,$$

where $d\sigma = (1+|z|^2)^{-2} dA(z)$ denotes the spherical area element.

Then there exists a normalized solution f of (B) such that $f \in \bigcap_{1 \le q < 2} W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ and that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

4.6.7 David

David [70] considered ACL embeddings $f : D \to \mathbb{C}$ with a complex dilatation μ which satisfies the exponential condition

$$|\{z \in D : |\mu(z)| > 1 - \varepsilon\}| \le C e^{-d/\varepsilon} \tag{4.6.4}$$

for all $\varepsilon \in (0, \varepsilon_0]$, for some $\varepsilon_0 \in (0,1]$, $C > 0$ and $d > 0$. Here, $|\{\cdots\}|$ denotes the Lebesgue measure of $\{\cdots\}$.

One of the main results in [70] (page 27) says that if μ is measurable in $\mathbb{C}$, and satisfies the exponential condition (4.6.4), then the Beltrami equation has a homeomorphic solution $f^\mu : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, which fixes the points 0, 1, and ∞. Furthermore, the mapping f^μ belongs to $\mathrm{W}^{1,s}_{\mathrm{loc}}$ for all $s < 2$. He also proves that if f is another homeomorphic solution and if $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{C})$, then $f = h \circ f^\mu$ for some conformal mapping h. David's proof of the existence follows the lines of Ahlfors [9] and Bojarski [44] involving a very fine analysis of singular integrals.

4.6.8 Tukia

In Tukia [254], (4.6.4) is replaced by the spherical exponential condition

$$\sigma\{z \in D : K(z) > t\} \le C e^{-\alpha t} \tag{4.6.5}$$

for all $t \ge T$, for some $T \ge 0$, $C > 0$ and $\alpha > 0$ and domains D in $\overline{\mathbb{C}}$. Here, $\sigma\{\cdots\}$ denotes the spherical area of $\{\cdots\}$. Tukia proves the existence of a global solution by using David's local solutions and the uniformization theorem. The latter is possible by David's uniqueness theorem. Tukia studies also convergence and compactness properties.

4.6.9 Ryazanov, Srebro and Yakubov

Ryazanov, Srebro, and Yakubov also obtained a series of results on the existence, uniqueness, and properties of ACL homeomorphic solutions for the degenerate Beltrami equations that are presented in Chaps. 5–8; see especially 8.5 for comparisons and also the works [210–219].

4.7 Modulus Inequalities

Our approach for deriving criteria for existence of solutions to the degenerate Beltrami equations is based mainly on the modulus techniques; see the original paper [12] and the recent book [165] which contain, in particular, the extended bibliography on the topic.

The modulus inequality (2.5.6) was extended and applied in [162] to orientation-preserving homeomorphisms $f : D \to \mathbb{R}^n, n \geq 2$, in $W^{1,n}_{\text{loc}}(D)$ with $K \in L^{n-1}_{\text{loc}}$; see also [164]. In [163], the modulus inequality was generalized to noninjective mappings. This inequality corresponds to the K_{I} modulus inequality (see [203]), which plays an important role in the quasiregular theory. A modulus inequality which corresponds to the K_O modulus inequality (see [203]) is also presented in [163]. It is shown in [163] that these two types of inequalities hold in the class of all mappings of finite length distortion. This class contains as a subclass all quasiregular mappings [163]. Later these and similar inequalities were extended in [125] to mappings of bounded distortion in $\mathbb{R}^n$ with K satisfying certain integrability conditions. These inequalities are used in [125] and in a series of papers in establishing the basic properties of mappings with finite distortion in all dimensions, such as openness, discreteness, Lusin's condition (N), and more.

Chapter 5
BMO- and FMO-Quasiconformal Mappings

In this chapter, the BMO-quasiconformal and BMO-quasiregular mappings in the plane are studied. This includes distortion, existence, uniqueness, representation, integrability, convergence and removability theorems, the reflection principle, boundary behavior, and mapping properties. We also study here the FMO-quasiconformal mappings. As shown by examples in Sect. 2.3.1, FMO is not a subclass of $\mathrm{BMO}_{\mathrm{loc}}$ and, furthermore, even not in L^p_{loc} for any $p > 1$. Thus, no earlier theory can be applied to the case.

5.1 Introduction

The plane BMO-quasiconformal and BMO-quasiregular mappings were first introduced and studied in the papers [210–212]. These maps are considered also in [17, 19] and [226]. For similar classes of mappings in higher dimensions, see the recent important papers [25, 115] and [116]. As in our works [99] and [210–223] as well as in [56, 57], the proof of existence is based here on approximation and extremal length methods, where the latter is mainly used for equicontinuity. BMO-qc and $\mathrm{BMO}_{\mathrm{loc}}$-qc mappings are closely related to mappings which were considered by David [70] and Tukia [254].

Recently, Brakalova and Jenkins [56] proved that given a measurable function μ in a domain D with $||\mu||_\infty \leq 1$, a μ-homeomorphism exists if the following two conditions hold:

$$\iint\limits_B \exp \frac{\frac{1}{1-|\mu|}}{1+\log\left(\frac{1}{1-|\mu|}\right)} \mathrm{d}A < \Phi_B \tag{5.1.1}$$

for every bounded measurable set B in D where Φ_B is a constant which depends on B, and

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$$\iint\limits_{\{|z|<R\}\cap D} \frac{1}{1-|\mu|}\,\mathrm{d}A = O(R^2), \quad R\to\infty. \tag{5.1.2}$$

We show in Sect. 5.4 that if $||\mu(z)||_\infty \le 1$ in a domain D, and a.e.

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \le Q(z), \quad Q\in \mathrm{BMO}_{\mathrm{loc}}, \tag{5.1.3}$$

then a μ-homeomorphism $f: D\to\mathbb{C}$ exists.

Locally, the conditions in (4.6.4) and (4.6.5) and (5.1.3) are equivalent and are stronger than (5.1.1). Thus, the existence of local homeomorphic solutions in the case where μ satisfies (4.6.5) or (5.1.3) follows from David's existence theorem, and locally, the existence theorem of Brakalova and Jenkins is stronger. We do not know if one can obtain a global homeomorphic solution if (5.1.1) holds only locally in D. A global homeomorphic solution can be obtained in our case (5.1.3) as in (1.4) in [254] by the uniformization theorem and David's uniqueness theorem for homeomorphic solutions [70], p. 55. We prefer, however, to give a direct proof of the existence of a global (as well as a local) homeomorphic solution using geometric methods which are also suitable for the study of BMO-qc mappings in higher dimensions. As in [56], our proof uses distortion lemmas (different from those in [56]) obtained from modulus estimates for ring domains. Our estimates are based on properties of BMO functions and on classical inequalities for path families, V(6.6) in [152] or [153], and 12.7 in [258], while those of [56] are based on a result from [198] on the modulus of the image of a circular ring domain under a μ-homeomorphism which is qc. Recently, Iwaniec and Martin [116] provided a proof of existence based on singular integral theory and harmonic analysis generalizing all previous cases mentioned above.

For a domain D in $\mathbb{C}$, let $\mathbf{D}(D)$, $\mathbf{T}(D)$, $\mathbf{BJ}(D)$, and $\mathbf{BMO\text{-}QC}(D)$ denote the classes of all ACL sense-preserving homeomorphisms $f: D\to\bar{\mathbb{C}}$ which satisfy a.e. in D the exponential conditions (4.6.4), (4.6.5), (5.1.1)–(5.1.2), and (5.1.3), respectively. Clearly, $\mathbf{D}(D)\subset\mathbf{T}(D)$ with equality when D is bounded. By [56], p. 89, $\mathbf{D}(D)\subset\mathbf{BJ}(D)$ and obviously $\mathbf{BJ}(D)\setminus\mathbf{BMO\text{-}QC}(D)\neq\emptyset$. We show in Sect. 5.2 that, globally, $\mathbf{D}$ and $\mathbf{T}$ are proper subclasses of $\mathbf{BMO\text{-}QC}$ and that for some domains, $\mathbf{BMO\text{-}QC}\setminus\mathbf{BJ}\neq\emptyset$. In case that D is a quasidisk then $\mathbf{Tu}(D)=\mathbf{RSY}(D)$, otherwise $\mathbf{Tu}(D)$, and hence $\mathbf{Da}(D)$, are proper subclasses of $\mathbf{RSY}(D)$. Each of the classes $\mathbf{BJ}(D)$ and $\mathbf{RSY}(D)$ contains functions which do not belong to the other class. It should be noted that **BMO-QC**, **D**, and **T** are qc invariant, while **BJ** does not have this property.

The **BMO-QC** setting has the advantage that it leads naturally to an extension of the qc-theory. In Sect. 5.3, we prove distortion lemmas. These are then combined with a known result about $Q(z)$-qc mappings for $Q\in\mathrm{L}^1_{\mathrm{loc}}$ [204, 205], yielding convergence lemmas, which lead in Sect. 5.4 to a short proof of the existence theorem mentioned above. A good approximation theorem is presented in Sect. 5.5, and various properties of BMO-qc and BMO-qr mappings are established in Sect. 5.6.

The rest of this chapter, Sects. 5.7–5.9, is devoted to Beltrami equations whose dilatations a.e. satisfy the condition (5.1.3) with $Q \in \text{FMO}$.

Throughout the chapter, $\mathbb{D}$ will denote the unit disk $|z| < 1$. For $a \in \mathbb{C}, \lambda > 1$ and $r > 0, B(a,r)$ denotes the disk $|z-a| < r$, and $\lambda B(a,r) = B(a,\lambda r)$. For $a,b \in \bar{\mathbb{C}}$ and $A \subset \bar{\mathbb{C}}, k(a,b)$ is the chordal distance between a and b and $k(A)$ is the chord diameter of A. For $A \subset \mathbb{C}, |A|$ denotes the Lebesgue measure of A, and for $A \subset \bar{\mathbb{C}}, \sigma(A)$ is the spherical area of A.

5.2 Inclusions, Integrability, and Group Properties

Proposition 5.1. *(i)* $\mathbf{T}(D) \subset \mathbf{BMO}\text{-}\mathbf{QC}(D)$.

(ii) If D is a quasidisk, then $\mathbf{T}(D) = \mathbf{BMO}\text{-}\ \mathbf{QC}(D)$.

(iii) For some domains $D, \mathbf{T}(D)$ *is a proper subset of* $\mathbf{BMO}\text{-}\mathbf{QC}(D)$.

(iv) For some domains $D, \mathbf{BMO}\text{-}\mathbf{QC}(D) \setminus \mathbf{BJ}(D) \neq \emptyset$.

Proof. (i) Given f in $\mathbf{T}(D)$, set $K(z) = 1$ in $\overline{\mathbb{C}} \setminus D$. For $t > T$ and $\beta = \frac{3}{4}\alpha$, let $\tau = \beta t, \eta = \mathrm{e}^{\tau}$ and

$$E_t = \{z \in \mathbb{C} : K(z) > t\}.$$

Then $E_t = \{z \in \mathbb{C} : \mathrm{e}^{\beta K(z)} > \eta\}$, and in view of (4.6.5), $\sigma(E_t) \leq C\eta^{-4/3}$. Hence,

$$\int_{E_t} \mathrm{e}^{\beta K(z)}\,\mathrm{d}\sigma(z) \leq C \int_{\mathrm{e}^{\beta T}}^{\infty} \eta^{-4/3}\,\mathrm{d}\eta \leq 3C\mathrm{e}^{-\beta T/3}.$$

Since $\mathrm{e}^{\beta K(z)}$ is bounded in $\bar{\mathbb{C}} \setminus E_t$, it follows that $\mathrm{e}^{\beta K(z)} \in L^1_{\sigma}$.

Let $q(z) = \mathrm{e}^{\alpha K(z)/2}$. Then $q(z) \in L^{3/2}_{\text{loc}}$. Let $Mq(z)$ denote the maximal function of $q(z)$, i.e.

$$Mq(z) = \sup_B q_B(z),$$

where the supremum is taken over all disks $B = B(z)$ centered at z. Then $Mq(z) \in L^{3/2}_{\text{loc}}$, and hence $\log Mq(z) \in \text{BMO}$, by [69], p. 249. Now set $Q(z) = \frac{1}{\beta}\log Mq(z)$. Since $\mathrm{e}^{\beta K(z)} \leq Mq(z)$, it follows that $K(z) \leq Q(z)$. This means that $f \in \mathbf{BMO}\text{-}\mathbf{QC}(D)$, which proves (i).

(ii) By the invariance of the given classes under quasiconformal mappings of the plane, we may assume without loss of generality that D is a disk. Then, by the known result of Jones [121], every $Q(z)$ in $\text{BMO}(\Delta)$ has an extension $\hat{Q}(z)$ which is in $\text{BMO}(\mathbb{C})$ with $||\hat{Q}||_* \leq C||Q||_*$, where C depends only on D. Hence, by [191], p. 7, any extension of $\hat{Q}$ on $\bar{\mathbb{C}}$ is in $\text{BMO}(\bar{\mathbb{C}})$. Then by John–Nirenberg's lemma [122] translated to the spherical metric, it follows that

$$\sigma\{z \in D : K(z) > t\} \leq \alpha\,\mathrm{e}^{-\beta t}|D|$$

for every $t > T$, i.e. $f \in \mathbf{T}(D)$.

(iii) Consider the mapping $f(z) = \frac{1}{2}x^2 + \mathrm{i}y$ in the half-strip $D = \{z = x + \mathrm{i}y : x > 1, |y| < 1\}$. Then $K(z) = x, z \in D$, belongs to $\mathrm{BMO}(D)$. So f is in **BMO-QC**(D). On the other hand, for $D_\mathrm{t} = \{z \in D : K(z) > t\}$,

$$\sigma(D_\mathrm{t}) = \iint\limits_{D_\mathrm{t}} \left(\frac{2}{1+|z|^2}\right)^2 \mathrm{d}m(z) \geq \frac{8}{9}\int\limits_t^\infty \frac{\mathrm{d}x}{x^4} = \frac{8}{27}t^{-3}.$$

Hence, $f \notin \mathbf{T}(D)$. This proves (iii).

(iv) Consider the mapping $g(\zeta) = \frac{1}{2}\xi^2|\zeta|^{-4} - \mathrm{i}\eta|\zeta|^{-2}, \zeta = \xi + \mathrm{i}\eta$ in the domain $D = B(1/2,1/2) \setminus [\bar{B}(\mathrm{i}/2,1/2) \cup \bar{B}(-\mathrm{i}/2,1/2)]$. Then $g(\zeta) = f(1/\zeta)$, where f is the function defined in the proof of (iii). Thus, $K_g(\zeta) = K_f(1/\zeta)$ is in $\mathrm{BMO}(D)$ by the Möbius invariance of BMO, and hence, $g \in$ **BMO-QC**(D). Using the change of variables $z = 1/\zeta$, an easy computation shows that g does not satisfy (5.1.1) in D, and (iv) follows. □

Proposition 5.2. *(i)* $\mathrm{BMO}_{\mathrm{loc}}$-*QC* $\subset W^{1,s}_{\mathrm{loc}}$ *for all* $s < 2$.
(ii) There are BMO-*qc mappings* f *which are not in* $W^{1,2}_{\mathrm{loc}}$.

Proof. (i) Suppose that $K = K_f \leq Q$ a.e., where $Q \in \mathrm{BMO}_{\mathrm{loc}}$ and f is a sense-preserving ACL homeomorphism. Since

$$|\partial f| + |\overline{\partial} f| = K^{1/2}(z) \cdot J^{1/2}(z) \leq Q^{1/2}(z)J^{1/2}(z) \text{ a.e.},$$

the assertion follows by Hölder's inequality and the fact that $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all finite $p \geq 1$ and the fact that $J \in L^1_{\mathrm{loc}}$; see Lemma 3.3 in [152], III3.4.

(ii) Consider the following map (see [70], p. 68):

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = R\mathrm{e}^{\mathrm{i}\theta} \quad \text{where } R = \left(\log\frac{1}{r}\right)^{-s}, \quad s > 0. \tag{5.2.1}$$

Let D_r and D_t be the radial and tangential stretching, respectively. Then $D_\mathrm{r} f = R'(r)$ and $D_\mathrm{t} f = R/r$ and, for small values of r, $K_f = D_\mathrm{t} f / D_\mathrm{r} f = s\log 1/r$. Hence, f is *BMO-qc* in a small disk around the origin. On the other hand, for small $\rho > 0$,

$$\iint\limits_{|z|\leq\rho} |D_\mathrm{t} f|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int\limits_0^\rho \frac{1}{r}\left(\log\frac{1}{r}\right)^{-2s} \mathrm{d}r.$$

Thus, for $0 < s \leq 1/2, f \notin \mathrm{W}^{1,2}_{\mathrm{loc}}$. Note that $f \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ for any $s > 1/2$. □

Proposition 5.3. *(i) There is a* BMO-*qc mapping* $f : D \to \mathbb{C}$ *such that* f^{-1} *is not in* BMO-*qc in* $f(D)$.
(ii) There are BMO-*qc mappings* $f : D \to \mathbb{C}$ *and* $g : f(D) \to \mathbb{C}$ *such that* $g \circ f$ *is not* BMO-*qc*.

Proof. (i) The map f given by (5.2.1) is in **BMO-QC** in some neighborhood of 0. Now, $f^{-1}:(R,\theta)\to(r,\theta)$ with $r=\mathrm{e}^{-R^{-1/s}}$ and $K_{f^{-1}}=1/(sR^{1/s})$. Since for large t,

$$|\{w:K_{f^{-1}}>t\}|=\pi(st)^{-2s},$$

it follows that $K_{f^{-1}}$ cannot be majorized by any BMO function, and hence, $f^{-1}\notin$ **BMO-QC**.

(ii) Let $f:(r,\theta)\to(R,\theta), R(r)=\mathrm{e}^{-2(\log r)^2}$, and $g:(R,\theta)\to(\rho,\theta),\rho(R)=(\log 1/R)^{-1}$; see [70], p. 68. Since f and g belong to **D** in some disk B about the origin, it follows by Proposition 5.1(i) that f and g are in **BMO-QC**(B). Set $F=g\circ f$. Then $K_F=8(\log 1/r)^3$ and, hence,

$$K_F(z)>t \quad \text{iff } |z|<\mathrm{e}^{\frac{-\sqrt[3]{t}}{2}}.$$

Therefore, $K_F(z)$ cannot be majorized by any function $Q(z)$ which belongs to BMO and consequently $F\notin$ **BMO-QC**. □

The following proposition shows why, for $p<\infty$, L^p-QC is not as good as **BMO-QC**:

Proposition 5.4. *For any $p\in[1,\infty)$, there exist:*

(i) A measurable function $\mu(z)$ in the unit disk $\mathbb{D}$ with $|\mu(z)|<1$ a.e. in $\mathbb{D}$ and $K(z)\in L^p(\mathbb{D})$ such that the Beltrami equation (4.6.4) has no homeomorphic solution in $\mathbb{D}$.

(ii) An L^p-qc mapping in the punctured disk $\mathbb{D}_0=\mathbb{D}\setminus\{0\}$ with a non-removable singularity at 0.

Proof. Let $p\in[1,\infty)$. Fix any α in $(0,2/p)$ and let $g:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ be given in polar coordinates r and ϑ by $g(r,\vartheta)=(1+r^\alpha,\vartheta)$. Then g maps the *punctured unit disk* $\mathbb{D}_0$ onto the annulus $A=\{z:1<|z|<2\}$ with dilatation:

$$K_g(r,\vartheta)=\max\left\{\frac{\alpha r^\alpha}{1+r^\alpha},\frac{1+r^\alpha}{\alpha r^\alpha}\right\},$$

which is bounded outside any neighborhood of $z=0$. Near $z=0$, $K_g=(1+r^\alpha)/\alpha r^\alpha$, i.e., p-integrable in the unit disk $\mathbb{D}$ due to the condition $p<2/\alpha$.

Let $\mu(z)=\bar{\partial}g/\partial g$ at $z\in\mathbb{D}_0$ and $\mu(0)=0$. Then $K=(1+|\mu(z)|)/(1-|\mu(z)|)$ is in $L^p(\mathbb{D})$. If (4.6.4) with this μ had a homeomorphic solution f in $\mathbb{D}$, then $f\circ g^{-1}$ would map A conformally onto the punctured domain $f(\mathbb{D})\setminus\{f(0)\}$. That is impossible. The conformality of $f\circ g^{-1}$ follows by Weyl's lemma from the fact that $\bar{\partial}(f\circ g^{-1})=0$ a.e. and that $f\circ g^{-1}$ is ACL since $g\in C^\infty$; see [9], p. 33.

Note that g is in L^p-QC in $0<|z|<1$ but 0 is a nonremovable singularity. □

5.3 Distortion Lemmas

In this section, we present distortion lemmas for qc mappings which are $Q(z)$-*qc* for a given function $Q(z)$. These lemmas will be needed in the proof of the existence theorem, Theorem 5.1. All results in this section are valid without the extra assumption that the mappings are qc, namely, that the dilatations are bounded, as long as $Q(z) \in \mathrm{BMO}_{\mathrm{loc}}$. This follows from the good approximation theorem, Theorem 5.3, which we prove in Sect. 5.5.

In the following lemma, we prove a property of a real-valued BMO function in a disk, which is needed in the proof of the distortion lemma. Thus, this lemma, first established in [210], plays a crucial role in proving equicontinuity.

Lemma 5.1. *For* $0 < t < \mathrm{e}^{-2}$, *let* $A(t) = \{z : t < |z| < \mathrm{e}^{-1}\}$. *Let* Q *be a nonnegative BMO function in* $\mathbb{D}$. *Then*

$$\eta(t) := \iint\limits_{A(t)} \frac{Q(z)\,\mathrm{d}m(z)}{|z|^2(\log|z|)^2} \leq c \log\log(1/t), \tag{5.3.1}$$

where c *is a constant which depends only on the average* Q_1 *of* Q *over* $|z| < \mathrm{e}^{-1}$ *and on the BMO norm* $||Q||_*$ *of* Q *in* $\mathbb{D}$.

Proof. Fix $t \in (0, \mathrm{e}^{-2})$. For $n = 1, 2, \ldots,$ let $A_n = \{z \in \mathbb{C} : t_{n+1} < |z| < t_n\}$, $B_n = \{z \in \mathbb{C} : |z| < t_n\}, t_n = \mathrm{e}^{-n}$, and Q_n be the mean value of $Q(z)$ in B_n. Now choose an integer N such that $t_{N+1} \leq t < t_N$. Then $A(t) \subset A(t_{N+1}) = \bigcup_{n=1}^{N+1} A_n$, and

$$\eta(t) \leq \iint\limits_{A(t_{N+1})} \frac{Q(z)}{|z|^2(\log|z|)^2}\mathrm{d}m(z) = S_1 + S_2, \tag{5.3.2}$$

where

$$S_1 = \sum_{n=1}^{N} \iint\limits_{A_n} \frac{Q(z) - Q_n}{|z|^2(\log|z|)^2}\mathrm{d}m(z) \tag{5.3.3}$$

and

$$S_2 = \sum_{n=1}^{N} Q_n \iint\limits_{A_n} \frac{\mathrm{d}m(z)}{|z|^2(\log|z|)^2}. \tag{5.3.4}$$

Since $A_n \subset B_n$, and for $z \in A_n, |z|^{-2} \leq \pi\mathrm{e}^2/|B_n|$ and $\log 1/|z| > n$, it follows that

$$|S_1| \leq \sum_{n=1}^{N} \iint\limits_{A_n} \frac{|Q(z) - Q_n|}{|z|^2(\log|z|)^2}\mathrm{d}m(z) \leq \pi \sum_{n=1}^{N} \frac{\mathrm{e}^2}{n^2} \left(\frac{1}{|B_n|} \iint\limits_{B_n} |Q(z) - Q_n|\mathrm{d}m(z) \right).$$

Hence,

$$|S_1| \leq 2\pi\mathrm{e}^2||Q||_*. \tag{5.3.5}$$

Now, note that

$$|Q_k - Q_{k-1}| = \frac{1}{|B_k|}\left|\iint_{B_k}(Q(z)-Q_{k-1})\mathrm{d}m(z)\right| \le \frac{1}{|B_k|}\iint_{B_k}|Q(z)-Q_{k-1}|\mathrm{d}m(z)$$
$$= \frac{\mathrm{e}^2}{|B_{k-1}|}\iint_{B_k}|Q(z)-Q_{k-1}|\mathrm{d}m(z) \le \frac{\mathrm{e}^2}{|B_{k-1}|}\iint_{B_{k-1}}|Q(z)-Q_{k-1}|\mathrm{d}m(z)$$
$$\le \mathrm{e}^2||Q||_*.$$

Thus, by the triangle inequality,

$$Q_n \le Q_1 + \sum_{k=2}^{n}|Q_k - Q_{k-1}| \le Q_1 + n\mathrm{e}^2||Q||_*, \tag{5.3.6}$$

and since

$$\iint_{A_n}\frac{\mathrm{d}m(z)}{|z|^2(\log|z|)^2} \le \frac{1}{n^2}\iint_{A_n}\frac{\mathrm{d}m(z)}{|z|^2} = \frac{2\pi}{n^2},$$

it follows by (5.3.4) that

$$S_2 \le 2\pi\sum_{n=1}^{N}\frac{Q_n}{n^2} \le 2\pi Q_1\sum_{n=1}^{N}\frac{1}{n^2} + 2\pi\mathrm{e}^2||Q||_*\sum_{1}^{N}\frac{1}{n}. \tag{5.3.7}$$

Finally, $\sum_{n=1}^{N}1/n^2$ is bounded, and $\sum_{n=1}^{N}1/n < 1+\log N < 1+\log\log 1/t$, thus (5.3.1) follows from (5.3.5) and (5.3.7). □

Lemma 5.2. *Let* $f:\mathbb{D}\to\overline{\mathbb{C}}\setminus\{a,b\}, a,b\in\overline{\mathbb{C}}, k(a,b)=\delta>0$ *be a qc mapping which is a* $Q(z)$*-qc for some* $Q\in \mathrm{BMO}(\mathbb{D})$ *such that* $f(0)=0$*. Then for* $|z|<\mathrm{e}^{-2}$,

$$|f(z)| \le C(\log 1/|z|)^{-\alpha},$$

where C *and* α *are positive constants which depend only on* $\mathbb{D}$*, on the BMO norm* $||Q||_*$ *of* Q *in* $\mathbb{D}$*, and on the average* Q_1 *of* Q *on* $|z|<\mathrm{e}^{-1}$.

Proof. Fix $t\in(0,\mathrm{e}^{-2})$ and let Γ_t and $A(t)$ be as in Lemma 5.1. Then

$$\rho(z) = \begin{cases} \frac{1}{(\log\log 1/t)|z|\log 1/|z|} & \text{if } z\in A(t) \\ 0 & \text{if } z\in\mathbb{C}\setminus A(t) \end{cases}$$

is admissible for Γ_t; therefore, by (2.5.6) and Lemma 5.1, we get

$$M(f\Gamma_t) \le \frac{c}{\log\log 1/t},$$

where c is the constant in Lemma 5.1. Next, let $L_t = L(t,f) = \max\{\,|f(z)| : |z| = t\}$, and let $a' = a'(\delta) > 0$ be such that $k(a',\infty) = \delta$, and let Γ' denote the family of paths joining $[-L_t, 0]$ and $[a',\infty]$. Then by a well-known property of the modulus of a path family (see [258], Theorem 6.4) and by the extremality of Teichmüller ring domains and the well-known lower bound for their modulus [see (1.2) and (2.10) in [152], II], we obtain

$$M(f\Gamma_t) \ge M(\Gamma') \ge \pi \log\left[4\left(\frac{L_t}{L_t + a'}\right)^{-1/2}\right].$$

Thus, for $|z| = t$,

$$|f(z)| \le L_t \le C(\log 1/|z|)^{-\alpha}$$

for some positive constants C and α which depend only on δ, on $||Q||_*$ and Q_1, as asserted. □

Corollary 5.1. *Given a domain D in $\mathbb{C}$, Q in* $\mathrm{BMO_{loc}}(D)$*, and distinct points a and b in $\mathbb{C}$, the family of all qc mappings from D into $\mathbb{C}\setminus\{a,b\}$ which are $Q(z)$-qc is normal.*

5.4 One Existence Theorem

Theorem 5.1. *Let D be a domain in $\mathbb{C}$ and let μ be a complex-valued measurable function in D with $|\mu(z)| < 1$ a.e. such that $K(z) \le Q(z)$ a.e. in D for some function $Q \in \mathrm{BMO_{loc}}(D)$. Then the Beltrami equation* (B) *has a homeomorphic ACL solution $f^\mu : D \to \mathbb{C}$ such that $(f^\mu)^{-1} \in W^{1,2}_{\mathrm{loc}}$.*

Proof. For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \le 1 - 1/n$ and 0 otherwise. Then there is a homeomorphic solution $f_n : D \to \mathbb{C}$ of the Beltrami equation (B) with μ_n instead of μ which fixes some points z_1 and z_2 in D; see, e.g., [9, 44] and [152]. Since all f_n are qc and $Q(z)$-qc, it follows by Corollary 5.1 that the sequence (f_n) has a subsequence f_{n_k} which converges locally uniformly to some mapping f in D. Thus, the conclusion follows by Corollaries 2.3 and 2.4 from Sect. 2.6. □

Remark 5.1. By Propositions 2.1, 2.4 and 2.5 from Sect. 2.4, $f^\mu \in W^{1,p}_{\mathrm{loc}}$ for all $p \in [1,2)$ has the N^{-1} property and f^μ is regular, i.e., differentiable with $J_{f^\mu}(z) > 0$ a.e.

5.5 Uniqueness and Approximation

It is not hard to see that by Stoilow's factorization theorem for open discrete mappings, one gets the following factorization property for BMO-qr mappings. Here, D is a domain in $\mathbb{C}$.

Proposition 5.5. *If g is BMO-qr in a domain D, then $g = h \circ \varphi$ where φ is BMO-qc in D with $\mu_\varphi = \mu_g$ a.e. and h is holomorphic in $\varphi(D)$.*

The following uniqueness theorem of David is used in the proof of some of the properties of BMO-qc and BMO-qr mappings which are presented in the sequel; see [70], p. 55.

Theorem 5.2. *Let $f_i, i = 1,2$ be μ-homeomorphisms in a domain D where μ satisfies David's condition (4.6.4). Then $f_2 \circ f_1^{-1}$ is conformal.*

Based on Proposition 5.5, Theorem 5.2, and the John–Nirenberg lemma, one obtains the following corollary:

Corollary 5.2. *Let φ and g be respectively a BMO-qc and a BMO-qr mapping in a domain D with $\mu_\varphi = \mu_g$ a.e. Then $g = h \circ \varphi$ for some holomorphic function h in $\varphi(D)$.*

5.5.1 *Good Approximation*

Let $f : D \to \bar{\mathbb{C}}$ be a $Q(z)$-*qr* mapping. We say that a sequence (f_n) of $Q(z)$-*qr* mappings in a domain D with complex dilatations $\mu_n, n = 1,2,\ldots$, is a *good approximation* of f, if the following three conditions hold (cf. [152], IV, 5.4):

(i) $||\mu_n||_\infty < 1$ for all n.
(ii) $f_n \to f$ locally uniformly in D.
(iii) $\mu_n \to \mu$ a.e. in D.

Theorem 5.3. *Let D be a domain in $\mathbb{C}$ and $f : D \to \mathbb{C}$ be a $Q(z)$-qr mapping with complex dilatation μ where $Q \in \mathrm{BMO_{loc}}(D)$. Then f has a good approximation.*

Proof. Fix two points z_1 and z_2 in D. For $n \in \mathbb{N}$, let $\mu_n(z) = \mu(z)$ if $|\mu_n(z)| < 1 - 1/n$ and 0 otherwise. Let g_n be a homeomorphic solution of (4.6.4) with μ_n instead of μ, which fixes z_1 and z_2. Since all g_n are qc and $Q(z)$-*qc* and fix z_1 and z_2, it follows by Corollary 5.1 that the sequence (g_n) has a subsequence, denoted again by (g_n), which converges locally uniformly in D to some mapping g in D. Then, by Corollary 2.3, g is $Q(z)$-*qc* and satisfies (4.6.4) a.e. By Corollary 5.2, $f = h \circ g$ for some holomorphic mapping $h : g(D) \to f(D)$. Then the sequence $f_n = h \circ g_n$ is a good approximation of f.

Based on Corollary 5.2 and Montel's normality criterion for analytic functions, one obtains by the good approximation theorem, Theorem 5.3, the following extension of Corollary 5.1: □

Corollary 5.3. *Given Q in $\mathrm{BMO}_{\mathrm{loc}}(D)$ and two distinct points a and b in $\mathbb{C}$, the family of all $Q(z)$-qr mappings from D into $\mathbb{C}\setminus\{a,b\}$ is normal.*

The following theorem can be considered as an extension of Corollary 2.3:

Theorem 5.4. *Let D be a domain in $\mathbb{C}$, $Q(z)\in\mathrm{BMO}_{\mathrm{loc}}(D)$, a and b two distinct points in $\mathbb{C}$, E a set in D which has a limit point in D and f_n a sequence of $Q(z)$-qr mappings from D into $\mathbb{C}\setminus\{a,b\}$. If $\{f_n\}$ has a limit at every point $e\in E$, then $\{f_n\}$ converges locally uniformly in D to a BMO-qr mapping in D. Moreover, ∂f_n and $\bar\partial f_n$ converge to ∂f and $\bar\partial f$, respectively, weakly in L^1_{loc}. If, in addition, $\mu_{f_n}\to\mu$ a.e., then $\bar\partial f=\mu\partial f$ a.e.*

Proof. It suffices to prove the statement for $D=\mathbb{D}$ and nonconstant f_n. By Proposition 5.5, $f_n=h_n\circ g_n$ where h_n is analytic and g_n is $Q(z)$-qc. In view of Picard's theorem and the Riemann mapping theorem, we may assume that $g_n(\mathbb{D})=\mathbb{D}$ and $g_n(0)=0, g_n(1)=1$, for all n; see extension properties in Sect. 5.6. Then $\{h_n\}$ is normal by Montel's theorem and $\{g_n\}$ is normal by Corollary 5.3. The result follows now by Vitali's condensation principle for analytic functions and Corollary 2.3; see also Corollary 5.2 and Theorem 5.3. □

5.6 Other Properties

Let D be a domain in $\mathbb{C}, E$ a free boundary arc in ∂D and $f:D\to\overline{\mathbb{C}}$ a (continuous) mapping. Then the *cluster set* $C(f,E)$ of f on E is a continuum, and if f is an embedding, then $C(f,E)\subset\partial f(D)$; cf. [68].

Theorem 5.5 (The reflection principle for BMO-qc mappings). *Let D be a domain in the upper half plane $\mathbb{H}_+=\{z\in\mathbb{C}:\mathrm{Im}\,z>0\}$, E a free boundary arc in ∂D and $f:D\to\mathbb{C}$ a BMO-qc mapping. Suppose that $E\subset\mathbb{R}$ and let $D^*=\{z\in\mathbb{C}:\bar z\in D\}$ and $\Omega=D\cup E\cup D^*$. If $C(f,E)\subset\mathbb{R}$ and $\mathrm{Re}\,f(z)>0$ for $z\in D$, then f has a BMO-qc extension to Ω.*

Proof. Since f is BMO-qc in D, there is a function Q in $\mathrm{BMO}(D)$ such that $K(z)\le Q(z)$ a.e. in D. Let $\hat Q$ be a symmetric extension of Q to Ω. Then $\hat Q\in\mathrm{BMO}(\Omega)$ and $||\hat Q||_*=||Q||_*$; see [191], p. 8. Let $\hat\mu:\Omega\to\mathbb{C}$ be a symmetric extension to Ω of the complex dilatation of f in D, and let $\hat K$ denote the corresponding dilatation. Then $\hat K(z)\le\hat Q(z)$ a.e. in Ω. Therefore, by Theorem 5.1, there exists a $\hat\mu$-homeomorphism $\hat f:\Omega\to\mathbb{C}$, and by Corollary 5.2, there is a conformal map h of $\hat f(D)$ such that $f(z)=h(\hat f(z))$ for all $z\in D$. Since $\hat f(E)$ is a Jordan arc in $\hat f(\Omega)$ and $C(h,\hat f(E))=C(f,E)\subset\mathbb{R}$, it follows that h has a homeomorphic extension to $\hat f(D)\cup\hat f(E)=\hat f(D\cup E)$, which is denoted again by h, and $f(z)=h(\hat f(z))$ for all $z\in D\cup E$.

This shows that f is homeomorphic in $D \cup E$ and hence that f is homeomorphic in Ω. Now, f is ACL in $D \cup D^*$, and since it is continuous on E, it is ACL in Ω. Evidently, the dilatation of f is symmetric with respect to E; hence, it coincides a.e. with $\hat{K}$ and it is thus dominated by $\hat{Q}$. Consequently, f is BMO-qc. □

Corollary 5.4. *Let D_1 and D_2 be domains in $\mathbb{C}$ and f a* BMO*-qc map of D_1 on D_2:*

(i) (Carathéodory's theorem) If D_1 and D_2 are Jordan domains, then f extends to a homeomorphism of $\overline{D}_1$ onto $\overline{D}_2$.
(ii) If, in addition, each ∂D_i is a quasicircle, then f has a BMO*-qc extension to $\overline{\mathbb{C}}$.*

Proof. Let φ_i be a conformal mapping of D_i onto $\mathbb{D}, i = 1,2$. Then $F = \varphi_2 \circ f \circ \varphi_1^{-1}$ is BMO-qc, and hence, by Theorem 5.5, has a homeomorphic extension to $\mathbb{D}$. Now, by Carathéodory's theorem, for $i = 1,2$, φ_i has a homeomorphic extension to ∂D_i, and the assertion follows. □

Theorem 5.6. *(i) Every BMO-qc mapping f in the punctured unit disk $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$ has a BMO-qc extension to $\mathbb{D}$.*
(ii) Every BMO-qc in $\mathbb{C}$ has a BMO-qc extension to $\overline{\mathbb{C}}$.

Proof. We prove (i). The proof of (ii) is similar. Let f be a BMO-qc map, and let μ denote its complex dilatation and K its dilatation. Then $K(z) \leq Q(z)$ a.e. for some function Q in BMO $(\mathbb{D}_0)$. Let $\hat{\mu}$ be any extension of μ on $\mathbb{D}, \hat{K}$ the corresponding dilatation and $\hat{Q}$ any extension of Q on $\mathbb{D}$. Then $\hat{Q} \in \text{BMO}(\mathbb{D})$ with $||\hat{Q}||_* = ||Q||_*$; see [191], p. 8. Clearly, $\hat{K} \leq \hat{Q}$ a.e. in $\mathbb{D}$. Hence, by Theorem 5.1, a BMO-*qc* mapping $\hat{f} : \mathbb{D} \to \mathbb{C}$ with dilatation $\hat{\mu}$ exists, and by Corollary 5.2, $f = h \circ (\hat{f}|\mathbb{D}_0)$ for some conformal mapping h of $\hat{f}(\mathbb{D}_0)$ onto $f(\mathbb{D}_0)$. Since h is conformal in $\hat{f}(\mathbb{D}_0)$ and $\hat{f}(\mathbb{D}_0) = \hat{f}(\mathbb{D}) \setminus \{f(0)\}$, it follows that h has a conformal extension H on $\hat{f}(\mathbb{D})$. Let $F = H \circ \hat{f}$. Then F is a continuous extension of f to $\mathbb{D}$, which belongs to BMO-qc. □

Several mapping problems in quasiconformal mapping theory are based on the fact that f^{-1} is qc whenever f is. This argument cannot be used in the BMO-qc theory. However, the following proposition holds:

Proposition 5.6. *(i) There are no* BMO*-qc mappings of a proper subdomain D of $\mathbb{C}$ onto $\mathbb{C}$.*
(ii) There are no BMO*-qc mappings of a nondegenerate doubly connected domain D onto a punctured disk.*

Proof. (i) The case where D is not simply connected is trivial since $\mathbb{C}$ is simply connected. If D is simply connected, we may assume, by the conformal invariance of BMO-QC, that $D = \mathbb{D}$. Suppose that a BMO-qc map f of $\mathbb{D}$ onto $\mathbb{C}$ exists. Let μ denote the dilatation of f and K its dilatation. Then $K(z) \leq Q(z)$ a.e. for some function Q in BMO$(\mathbb{D})$. Let $\hat{\mu}$ be a symmetric extension of μ to $\mathbb{C}$. Then the corresponding dilatation function $\hat{K}$ is dominated a.e. by a symmetric extension $\hat{Q}$ of Q, $\hat{Q} \in \text{BMO}(\overline{\mathbb{C}})$, and hence, by Theorem 5.1, there exists a BMO-qc map g of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ with the complex dilatation $\mu = \hat{\mu}$ and with

$g(\infty)=\infty$. Then $g(\mathbb{D})$ is a bounded domain in $\mathbb{C}$. By Corollary 5.2, $g|\mathbb{D}=h\circ f$ for some conformal mapping h of $f(\mathbb{D})$ onto $g(\mathbb{D})$. Thus, $f(D)$ is a proper subset of $\mathbb{C}$.

(ii) As above, we may assume that D is an annulus and that f exists. Extend $\mu_f(z)=f_{\bar{z}}/f_z$ symmetrically with respect to the boundary components of D. A repeated extension yields a complex dilatation defined a.e. in $\mathbb{C}$. Repeating the corresponding arguments as in (i), we arrive at the contradiction. □

Theorem 5.7 (Picard's theorems). *(i) Let f be a* BMO-*qr map of the punctured disk $\mathbb{D}_0$ into $\overline{\mathbb{C}}$. If f omits at least three values in $\overline{\mathbb{C}}$, then f has a* BMO-*qr extension to $\mathbb{D}$.*

(ii) Let f be a BMO-*qr map of $\mathbb{C}$ into $\overline{\mathbb{C}}$. If f omits at least three values in $\overline{\mathbb{C}}$, then f is a constant.*

Proof. As in the analytic case, it suffices to prove (i). Since f is a BMO-*qr*, it has the representation $f=h\circ g$, where g is BMO-*qc* and h is holomorphic. By Theorem 5.6 (i), g has a BMO-*qc* extension G to Δ. Hence, $G(0)$ is an isolated singularity of h. Now, since f omits at least three values in $\overline{\mathbb{C}}$, so does h, and hence, by Picard's theorem, h has a holomorphic extension H on $G(\mathbb{D})$, and thus $F=H\circ G$ is a BMO-*qr* extension of f. □

Theorem 5.8 (The reflection principle for BMO-qr mappings). *Let D, D^*, E, and Ω be as in Theorem 5.5. If $f: D\to\overline{\mathbb{C}}$ is* BMO-*qr and $C(f,E)\subset\mathbb{C}$, then f has a continuous extension to $D\cup E$, denoted again by f, and*

$$F(z)=\begin{cases} f(z), & \text{if } z\in D\cup E\\ \overline{f(\bar{z})}, & \text{if } z\in D^*\end{cases}$$

is BMO-*qr.*

Proof. First, we prove the theorem when D is a simply connected domain. Since f is BMO-qr, it has a representation $f=h\circ g$, where g is a BMO-qc map and h is holomorphic in $D'=g(D)$. By Proposition 5.6 (i), $D'\neq\mathbb{C}$. We thus may assume that D' is the upper half plane $\mathbb{H}_+$. Otherwise, replace g by $\varphi\circ g$ and h by $h\circ\varphi^{-1}$, where φ is a conformal map of D' onto $\mathbb{H}_+$.

By Theorem 5.5, g has a continuous extension onto $D\cup E$, denoted again by g, and the symmetric extension of g on Ω denoted by G is BMO-*qc* in Ω. Let $E'=g(E)$. Since g is a homeomorphism on $D\cup E$ and $h=f\circ g^{-1}$, it follows that $C(h,E')=C(f,E)\subset\mathbb{R}$. Hence, by the reflection principle for holomorphic functions, we may conclude that h has a continuous extension on E', which is denoted again by h and that the symmetric extension

$$H(z)=\begin{cases} h(z), & \text{if } z\in g(D)\cup E'\\ \overline{h(\bar{z})}, & \text{if } z\in g(D)^*\end{cases}$$

is holomorphic in $\Omega' = g(D) \cup E \cup g(D)^*$. Therefore, $G \circ H$ is BMO-qr. Now, H and G preserve symmetry, hence so does $G \circ H$, and $f = h \circ g$ on $D \cup E$; it follows that F coincides with $G \circ H$, and the proof is complete in the case when D is simply connected.

If D is not simply connected, we first choose a simply connected domain U which is symmetric with respect to $\mathbb{R}$, such that $E \subset U$ and $U \cap \mathbb{H}_+ \subset D$. Then $f|U \cap \mathbb{H}_+$ has a BMO-qr symmetric extension on U, which is denoted again by f, and hence, F is BMO-qr. □

Remark 5.2. Recently, Sastry in [226] provided necessary and sufficient conditions for a BMO-qc extension from $\mathbb{R}$ to the half plane H_+. Her conditions are of a form similar to Ahlfors–Beurling's, where the upper and lower bounds depend on the BMO norm and certain averages of a given BMO boundary function, respectively, of a BMO majorant Q of a given BMO-qc map of H_+.

One can show that none of the results in this section are valid for $\mathrm{BMO}_{\mathrm{loc}}$-qc mappings and $\mathrm{BMO}_{\mathrm{loc}}$-qr mappings.

5.7 The Main Lemma on FMO

Below we use the following notation:

$$A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\}. \tag{5.7.1}$$

Lemma 5.3. *Let $\varphi : D \to \mathbb{R}$ be a nonnegative function with finite mean oscillation at $0 \in D$ and integrable in $B(\mathrm{e}^{-1}) \subset D$. Then, for $\varepsilon \in (0, \mathrm{e}^{-e})$,*

$$\int\limits_{A(\varepsilon, \mathrm{e}^{-1})} \frac{\varphi(z)\,\mathrm{d}m(z)}{\left(|z| \log \frac{1}{|z|}\right)^2} \leq C \cdot \log\log \frac{1}{\varepsilon}, \tag{5.7.2}$$

where

$$C = 2\pi(2\varphi_0 + 3\,\mathrm{e}^2 d_0), \tag{5.7.3}$$

φ_0 is the mean value of φ over the disk $B(\mathrm{e}^{-1})$ and d_0 is the maximal dispersion of φ in the disk $B(\mathrm{e}^{-1})$.

The lemma plays a central role in the proof of equicontinuity. Versions of this lemma were first established for BMO functions in [210] and then for FMO functions in [113]. An n-dimensional version of the lemma for BMO functions was established in [162]. The proof here is similar to the one in [210] and is presented for the sake of completeness.

Proof. Let $0 < \varepsilon < \mathrm{e}^{-e}, \varepsilon_k = \mathrm{e}^{-k}, A_k = \{z \in \mathbb{C}^n : \varepsilon_{k+1} \leq |z| < \varepsilon_k\}$, and $B_k = B(\varepsilon_k)$, and let φ_k be the mean value of $\varphi(z)$ over $B_k, k = 1, 2 \ldots$. Take a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$. Then $A(\varepsilon, \mathrm{e}^{-1}) \subset A(\varepsilon) = \bigcup_{k=1}^{N} A_k$ and

$$\eta(\varepsilon) = \int\limits_{A(\varepsilon)} \varphi(z)\, \alpha(|z|)\, \mathrm{d}m(z) \leq |S_1| + S_2,$$

where

$$\alpha(t) = (t \log 1/t)^{-2},$$

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int\limits_{A_k} (\varphi(z) - \varphi_k)\, \alpha(|z|) \mathrm{d}m(z),$$

and

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int\limits_{A_k} \alpha(|z|)\, \mathrm{d}m(z).$$

Since $A_k \subset B_k, |z|^{-2} \leq \pi \mathrm{e}^2/|B_k|$ for $z \in A_k$ and $\log 1/|z| > k$ in A_k, then

$$|S_1| \leq \pi \mathrm{e}^2 d_0 \sum_{k=1}^{N} \frac{1}{k^2} < 2\pi \mathrm{e}^2 d_0,$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^2} < \int\limits_{1}^{\infty} \frac{\mathrm{d}t}{t^2} = 1.$$

Now,

$$\int\limits_{A_k} \alpha(|z|) \mathrm{d}m(z) \leq \frac{1}{k^2} \int\limits_{A_k} \frac{\mathrm{d}m(z)}{|z|^2} = \frac{2\pi}{k^2}.$$

Moreover,

$$|\varphi_k - \varphi_{k-1}| = \frac{1}{|B_k|} \left| \int\limits_{B_k} (\varphi(z) - \varphi_{k-1}) \mathrm{d}m(z) \right|$$
$$\leq \frac{\mathrm{e}^2}{|B_{k-1}|} \int\limits_{B_{k-1}} |\varphi(z) - \varphi_{k-1}| \mathrm{d}m(z) \leq \mathrm{e}^2 d_0,$$

and by the triangle inequality, for $k \geq 2$,

$$\varphi_k = |\varphi_k| \leq \varphi_1 + \sum_{l=2}^{k} |\varphi_l - \varphi_{l-1}| \leq \varphi_1 + k\mathrm{e}^2 d_0 = \varphi_0 + k\mathrm{e}^2 d_0.$$

Hence,

$$S_2 = |S_2| \leq 2\pi \sum_{k=1}^{N} \frac{\varphi_k}{k^2} \leq 4\pi \varphi_0 + 2\pi \mathrm{e}^2 d_0 \sum_{k=1}^{N} \frac{1}{k}.$$

But

$$\sum_{k=2}^{N}\frac{1}{k}<\int_{1}^{N}\frac{\mathrm{d}t}{t}=\log N$$

and, for $\varepsilon<\varepsilon_N$,

$$N=\log\frac{1}{\varepsilon_N}<\log\frac{1}{\varepsilon}.$$

Consequently,

$$\sum_{k=1}^{N}\frac{1}{k}<1+\log\log\frac{1}{\varepsilon},$$

and thus, for

$$\varepsilon\in(0,\mathrm{e}^{-e}),\ \eta(\varepsilon)\leq 2\pi\left(\mathrm{e}^2 d_0+2\cdot\frac{\mathrm{e}^2 d_0+\varphi_0}{\log\log\frac{1}{\varepsilon}}\right)\cdot\log\log\tfrac{1}{\varepsilon}\leq C\cdot\log\log\tfrac{1}{\varepsilon}. \qquad \square$$

Remark 5.3. The concept of finite mean oscillation can be extended to infinity in the standard way. Namely, given a domain D in the extended complex plane $\overline{\mathbb{C}}$, $\infty\in D$, and a function $\varphi: D\to\mathbb{R}$, we say that φ has finite mean oscillation at ∞ if the function $\varphi^*(z)=\varphi(1/\overline{z})$ has finite mean oscillation at 0. Clearly, by the change of variables $z\rightarrowtail 1/\overline{z}$, the latter is equivalent to the condition

$$\int\limits_{|z|\geq R}|\varphi(z)-\overline{\varphi}_R|\frac{\mathrm{d}m(z)}{|z|^4}=O\left(\frac{1}{R^2}\right), \tag{5.7.4}$$

where

$$\overline{\varphi}_R=\frac{R^2}{\pi}\int\limits_{|z|\geq R}\varphi(z)\frac{\mathrm{d}m(z)}{|z|^4}. \tag{5.7.5}$$

5.8 Estimate of Distortion

For points $a,b\in\mathbb{C}$ and a set $E\subset\overline{\mathbb{C}}$, the chord distance and chord diameter will be denoted by $s(a,b)$ and $\delta(E)$; see (2.5.4) and (2.5.5), respectively. Given a domain $D\subset\mathbb{C}$, a measurable function $Q:D\to[1,\infty]$ and a number $\Delta>0$, let $\mathfrak{F}_Q^{\Delta}$ denote the class of quasiconformal mappings $f:D\to\overline{\mathbb{C}}$ such that

$$\delta(\overline{\mathbb{C}}\setminus f(D))\geq\Delta \tag{5.8.1}$$

and such that

$$K_f(z)\leq Q(z)\quad \text{a.e. in } D. \tag{5.8.2}$$

This means, in particular, that f is a sense-preserving ACL homeomorphism and, in addition to (5.8.2), $K_f \in L^\infty(D)$.

Lemma 5.4. *Let D be a domain in $\mathbb{C}$ with $\overline{B(1/e)} \subset D$ and $f: D \to \mathbb{C}$ a qc mapping. If $f \in f \in \mathfrak{F}_Q^\Delta$ for some $\Delta > 0$ and some function $Q: B(1/e) \to [1,\infty]$ which is integrable in $B(1/e)$ and which is of finite mean oscillation at 0, then*

$$s(f(z), f(0)) \leq \alpha_0 \cdot \left(\log \frac{1}{|z|}\right)^{-\beta_0} \tag{5.8.3}$$

for every point $z \in B(\mathrm{e}^{-e})$, where

$$\alpha_0 = \frac{32}{\Delta}, \tag{5.8.4}$$

$$\beta_0 = (2q_0 + 3\mathrm{e}^2 d_0)^{-1}, \tag{5.8.5}$$

q_0 is the mean value of $Q(z)$ in $B(1/e)$ and d_0 is the maximal dispersion of $Q(z)$ in $B(1/e)$.

Proof. Let Γ_ε be a family of all paths joining the circles $S_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon\}$ and $S_0 = \{z \in \mathbb{C} : |z| = \mathrm{e}^{-1}\}$ in the ring $A_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z| < \mathrm{e}^{-1}\}$. Then the function:

$$\rho_\varepsilon(z) = \begin{cases} \dfrac{a_\varepsilon}{\frac{1}{z}\log\frac{1}{|z|}}, & \text{if } z \in A_\varepsilon, \\ 0, & \text{if } z \in \mathbb{C}\setminus A_\varepsilon, \end{cases} \tag{5.8.6}$$

where

$$a_\varepsilon = \left(\log\log\frac{1}{\varepsilon}\right)^{-1}, \tag{5.8.7}$$

is admissible for Γ_ε, and hence, by (2.5.6),

$$M(f\Gamma_\varepsilon) \leq \int_D Q(z) \cdot \rho_\varepsilon^2(|z|)\mathrm{d}m(z). \tag{5.8.8}$$

Now, $Q \in \mathrm{FMO}$ at 0, and thus, by Lemma 5.3, (5.8.6) and (5.8.7),

$$\int_D Q(z) \cdot \rho_\varepsilon^2(|z|)\mathrm{d}m(z) \leq C \cdot a_\varepsilon \tag{5.8.9}$$

and hence

$$M(f\Gamma_\varepsilon) \leq \frac{C}{\log\log\frac{1}{\varepsilon}}, \tag{5.8.10}$$

where the constant C is as in Lemma 5.3.

Note that $\overline{\mathbb{C}}\backslash fA_\varepsilon$ has exactly two components because f is a homeomorphism and A_ε is a doubly connected domain. Denote by Γ_ε^* the family of all paths in $\overline{\mathbb{C}}$ joining the sets fS_ε and fS_0. Then

$$M(\Gamma_\varepsilon^*) = M(f\Gamma_\varepsilon). \tag{5.8.11}$$

Indeed, on one hand $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$ and hence $M(f\Gamma_\varepsilon) \le M(\Gamma_\varepsilon^*)$, and on the other hand $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$, i.e., every path in Γ_ε^* contains a subpath which belongs to $f\Gamma_\varepsilon$, and hence, $M(f\Gamma_\varepsilon) \ge M(\Gamma_\varepsilon^*)$; see, e.g., Theorem 1(c) in [85]. Finally, by choosing $\varepsilon = |z|$ and invoking Lemma 2.13, we obtain (5.8.3). □

Corollary 5.5. *Let D be a domain in $\mathbb{C}$, $B(z_0,r_0)$ a disk in $D, f : D \to \overline{\mathbb{C}}$ a qc mapping which belongs to $\mathfrak{F}_Q^\Delta$ for some $\Delta > 0$ and some function $Q : B(z_0,r_0) \to [1,\infty]$ which is integrable in $B(z_0,r_0)$. If $Q(z)$ is of finite mean oscillation at z_0, then*

$$s(f(z), f(z_0)) \le \alpha_0 \cdot \left(\log \frac{er_0}{|z-z_0|}\right)^{-\beta_0} \tag{5.8.12}$$

for every $z \in B(z_0, \mathrm{e}^{1-e} r_0)$, where α_0 and β_0 are as in (5.8.4) and (5.8.5), q_0 is the mean value of the function $Q(z)$ over the disk $B(z_0,r_0)$, and d_0 is the maximal dispersion of $Q(z)$ in $B(z_0,r_0)$.

Indeed, the mean value and the dispersion of a function over disks are invariant under linear transformations of the independent variable; thus, (5.8.12) follows from Lemma 5.4 by applying the transformation $z \rightarrowtail (z-z_0)/er_0$.

5.9 Further Existence Theorems

Based on the distortion estimates of Sect. 5.8, we obtain by a standard approximation process the following existence theorem:

Theorem 5.9. *Let D be a domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. If*

$$K_\mu(z) \le Q(z) \quad \text{a.e. in } D \tag{5.9.1}$$

for some FMO *function $Q : D \to [1,\infty]$, then the Beltrami equation* (B) *has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution $f_\mu : D \to \mathbb{C}$ with $f_\mu^{-1} \in W^{1,2}_{\mathrm{loc}}$.*

Proof. Fix points z_1 and z_2 in D. For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \le 1 - 1/n$ and 0 otherwise. Then $\|\mu_n\|_\infty < 1$, and thus, by the classical existence theorem, the Beltrami equation (B) with μ_n instead of μ has a homeomorphic ACL solution $f_n : D \to \mathbb{C}$ which fixes prescribed z_1 and z_2 in D; see, e.g., [9, 44] and [152]. By Corollary 5.5, the sequence f_n is equicontinuous, and hence, by the Arzela–Ascoli theorem (see, e.g., [73], p. 382, and [74], p. 267), it has a subsequence, denoted again by f_n, which converges locally uniformly to some nonconstant mapping f in D. Then f is a homeomorphic ACL solution of (B) by Theorem 2.1 and Corollary 2.3 from Sect. 2.6. Since $Q \in \mathrm{FMO}$, $Q \in L^1_{\mathrm{loc}}$, and hence $K_\mu \in L^1_{\mathrm{loc}}$. Therefore, f belongs to $W^{1,1}_{\mathrm{loc}}$ by Proposition 2.4. The inclusion $f^{-1} \in W^{1,2}_{\mathrm{loc}}$ follows by Corollary 2.4 from Sect. 2.6. □

Remark 5.4. By Proposition 2.5 from Sect. 2.4, f_μ has the N^{-1} property and f_μ is regular, i.e., differentiable with $J_{f^\mu}(z) > 0$ a.e.

Corollary 5.6. *If $K_\mu(z) \le Q(z)$ a.e. and every point $z \in D$ is a Lebesgue point for $Q(z)$, then the Beltrami equation* (B) *has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution f_μ with $f_\mu^{-1} \in W^{1,2}_{\mathrm{loc}}$.*

Corollary 5.7. *If, for every point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon \to 0}} \;\; ⨍_{B(z_0,\varepsilon)} K_\mu(z)\, \mathrm{d}m(z) < \infty, \tag{5.9.2}$$

then the Beltrami equation (B) *has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution f_μ with $J_{f_\mu}(z) > 0$ a.e.*

Remark 5.5. (1) Note that if $f : \mathbb{C} \to \mathbb{C}$ is $Q(z)$-qc with $Q \in \mathrm{BMO}$, then f is surjective and extends to a BMO-*qc* mapping of $\overline{\mathbb{C}}$ onto itself because isolated singularities are removable for BMO functions; see, e.g., [211]. However, a $Q(z)$-qc mapping f need not be surjective if the condition $Q \in \mathrm{BMO}$ is replaced by the weaker condition $Q \in \mathrm{FMO}$. For instance, the dilatation of every diffeomorphism of $\mathbb{C}$ onto the unit disk Δ is continuous and hence belongs to FMO, but $f : \mathbb{C} \to \mathbb{C}$ is not surjective. Note that f^{-1} is a diffeomorphism of Δ onto $\mathbb{C}$, and its dilatation is also in FMO.

(2) In view of Lemma 5.4, Theorem 5.9 extends to the case where $\infty \in D \subset \overline{\mathbb{C}}$ if the condition that $Q(z)$ has finite mean oscillation at ∞ is added; see Remark 5.3. In this case, there exists a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution $f = f_\mu$ in D with $f(\infty) = \infty$ and $f_\mu^{-1} \in W^{1,2}_{\mathrm{loc}}$. Here $f \in W^{1,1}_{\mathrm{loc}}$ in D means that $f \in W^{1,1}_{\mathrm{loc}}$ in $D \setminus \{\infty\}$ and that $f^*(z) = 1/\overline{f(1/\overline{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W^{1,2}_{\mathrm{loc}}$ has a similar meaning. Consequently, if the condition (5.9.2) holds at every point $z_0 \in D \setminus \{\infty\}$ and if

$$\int\limits_{|z|>R} K_\mu(z) \frac{\mathrm{d}m(z)}{|z|^4} = O\left(\frac{1}{R^2}\right) \quad \text{as } R \to \infty, \tag{5.9.3}$$

then the Beltrami equation (B) has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution f_μ with $f_\mu^{-1} \in W^{1,2}_{\mathrm{loc}}$.

(3) In view of the dilatation estimate in Lemma 5.4 and Corollary 5.5, the condition (5.9.1) in Theorem 5.9 can be localized, yielding the following corollaries:

Corollary 5.8. *Let D be a domain in $\mathbb{C}$ and $\mu : D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. If for every point $z_0 \in D$, there exist a disk $B(z_0, r_0) \subset D$ and a function $Q_{z_0} : B(z_0, r_0) \to [1, \infty]$ which is of finite mean oscillation at z_0 such that $K_\mu(z) \le Q_{z_0}(z)$ for a.e. $z \in B(z_0, r_0)$, then the Beltrami equation* (B) *has a homeomorphic $W^{1,1}_{\text{loc}}$ solution $f_\mu : D \to \mathbb{C}$ with $f_\mu^{-1} \in W^{1,2}_{\text{loc}}$.*

Corollary 5.9. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If*

$$K_\mu(z) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \to z_0 \tag{5.9.4}$$

for every point $z_0 \in \mathbb{C}$ and

$$K_\mu(z) = O(\log |z|) \quad \text{as } z \to \infty, \tag{5.9.5}$$

then the Beltrami equation (B) *has a homeomorphic $W^{1,1}_{\text{loc}}$ solution $f_\mu : \mathbb{C} \to \mathbb{C}$ with $f(\mathbb{C}) = \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{\text{loc}}$.*

Chapter 6
Ring Q-Homeomorphisms at Boundary Points

In this chapter, we introduce and study plane ring Q-homeomorphisms. This study is then applied to deriving general principles on the existence of strong ring solutions to the Beltrami equation extending and strengthening earlier results; see the next chapter.

6.1 Introduction

Basically, there are three definitions of quasiconformality: analytic, geometric, and metric. They are equivalent with the same parameter of quasiconformality K. By the analytic definition, a homeomorphism $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, is *K-quasiconformal*, abbreviated as *K-qc*, if f is ACL and

$$\operatorname{ess\,sup}\, K_\mu(z) = K < \infty, \tag{6.1.1}$$

where μ is the complex dilatation of f and

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{6.1.2}$$

is its dilatation; see Sects. 2.4 and 3.1 for details. According to the equivalent geometric definition (see, e.g., [258]), f is K-quasiconformal if

$$\sup \frac{M(f\Gamma)}{M(\Gamma)} = K < \infty, \tag{6.1.3}$$

where the supremum is taken over all path families Γ in D with modulus $M(\Gamma) \neq 0$. It was noted by Ahlfors and Gehring that the supremum in (6.1.3) can be taken over special families yielding the same bound K. In particular, by [89], one may restrict to families of paths connecting the boundary components of rings in D.

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Given a measurable function $Q : D \to [1,\infty]$, we say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a Q-*homeomorphism* if

$$M(f\Gamma) \leq \int\limits_D Q(z) \cdot \rho^2(z)\,\mathrm{d}m(z) \tag{6.1.4}$$

holds for every path family Γ in D and each $\rho \in \operatorname{adm}\Gamma$. This term was introduced in [162] (see also [164] and [163] and the monograph [165]), and the inequality was used in [210] and [211] as a basic tool in studying BMO-qc mappings.

Probably, the inequality

$$M(f\Gamma) \leq \int\limits_D K_\mu(z) \cdot \rho^2(z)\,\mathrm{d}m(z) \tag{6.1.5}$$

for $\mu = \mu_f$ was first obtained in [153], p. 221, (see also [152]) for quasiconformal mappings. Note that K_μ cannot be replaced by a smaller function in (6.1.5) unless one restricts either to special families Γ or to special $\rho \in \operatorname{adm}\Gamma$. In Sect. 7.2, (6.1.5) is improved for special Γ and ρ and then used for deriving new criteria for the existence of homeomorphic solutions of the Beltrami equation (B).

Motivated by the ring definition of quasiconformality in [89], we introduced the following notion that localizes and extends the notion of a Q-homeomorphism; see [220]. Let D be a domain in $\overline{\mathbb{C}}$, $z_0 \in \overline{D} \setminus \{\infty\}$, $D(z_0,r_0) = \{z \in D : |z-z_0| < r_0\}$, $r_0 \leq \sup_{z\in D}|z-z_0|$, and let $Q : D(z_0,r_0) \to [0,\infty]$ be a measurable function. We say that a homeomorphism f of D into $\overline{\mathbb{C}}$ is called a *ring Q-homeomorphism at the point* z_0 if

$$M(\Delta(fC_0, fC_1, fD)) \ \leq \int\limits_{A\cap D} Q(z) \cdot \eta^2(|z-z_0|)\,\mathrm{d}m(z) \tag{6.1.6}$$

for every ring $A = A(z_0,r_1,r_2) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}, \quad 0 < r_1 < r_2 < r_0$, and for every continua C_0 and C_1 in D which belong to the different components of the complement to the ring A in $\overline{\mathbb{C}}$, containing z_0 and ∞, respectively, and for every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,\mathrm{d}r \ = \ 1. \tag{6.1.7}$$

Note that the function Q in this definition can be always extended in a measurable way to D or even to $\overline{\mathbb{C}}$, say by zero.

The notion of ring Q-homeomorphism can be extended to ∞ in the standard way. More precisely, given a domain D in $\overline{\mathbb{C}}$, $\infty \in \overline{D}$, $D(\infty,r_0) = \{z \in D : |z| > r_0\}$, $r_0 \geq \inf_{z\in D}\ |z|$, a measurable function $Q : D(\infty,r_0) \to [0,\infty]$, we say that a homeomorphism f of D into $\overline{\mathbb{C}}$ is called a *ring Q-homeomorphism at* ∞, if the homeomorphism $f_* = f \circ j$ is a Q_*-homeomorphism at 0, where $Q_* = Q \circ j$ and $j(z) = z/|z|^2$ is the inversion with respect to the unit circle $|z| = 1$ in $\overline{\mathbb{C}}$, $j(\infty) = 0$ and $j(0) = \infty$.

Given a measurable function $Q : D \to [0,\infty]$, we also say that a homeomorphism f of D into $\overline{\mathbb{C}}$ is called a *ring Q-homeomorphism* if it is a ring Q-homeomorphism at every point $z_0 \in \overline{D}$.

Note that every Q-homeomorphism $f : D \to \overline{\mathbb{C}}$ is a ring Q-homeomorphism at each point $z_0 \in \overline{D}$. In the next section, we give other conditions on f which force it to be a ring Q-homeomorphism.

A point $z \in \mathbb{C}$ is called a *regular point* for a mapping $f : D \to \mathbb{C}$ if f is differentiable at z and $J_f(z) \neq 0$. Given $\omega \in \mathbb{C}$, $|\omega| = 1$, the *derivative in the direction* ω of the mapping f at the point z is

$$\partial_\omega f(z) \;=\; \lim_{t\to+0} \frac{f(z+t\cdot\omega)-f(z)}{t}\,. \tag{6.1.8}$$

The *radial direction* at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is

$$\omega_0 \;=\; \omega_0(z,z_0) \;=\; \frac{z-z_0}{|z-z_0|}\,. \tag{6.1.9}$$

The *radial dilatation* of f at z with respect to z_0 is defined by

$$K^r(z,z_0,f) \;=\; \frac{|J_f(z)|}{|\partial_r^{z_0} f(z)|^2} \tag{6.1.10}$$

and the *tangential dilatation* by

$$K^T(z,z_0,f) \;=\; \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|}, \tag{6.1.11}$$

where $\partial_r^{z_0} f(z)$ is the derivative of f at z in the direction ω_0 and $\partial_T^{z_0} f(z)$ in $\tau = \mathrm{i}\omega_0$, respectively.

Note that if z is a regular point of f and $|\mu(z)| < 1$, $\mu(z) = f_{\bar z}/f_z$, then

$$K^r(z,z_0,f) \;=\; K^r_\mu(z,z_0) \tag{6.1.12}$$

and

$$K^T(z,z_0,f) \;=\; K^T_\mu(z,z_0), \tag{6.1.13}$$

where

$$K^T_\mu(z,z_0) \;=\; \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{6.1.14}$$

and

$$K^r_\mu(z,z_0) = \frac{1-|\mu(z)|^2}{\left|1+\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}\,; \tag{6.1.15}$$

cf. [99, 147] and [198]. Indeed, the equalities (6.1.12) and (6.1.13) follow directly from the calculations

$$\partial_r f \;=\; \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \;+\; \frac{\partial f}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial r} \;=\; \frac{z-z_0}{|z-z_0|}\cdot f_z \;+\; \frac{\overline{z-z_0}}{|z-z_0|}\cdot f_{\bar z}, \tag{6.1.16}$$

where $r=|z-z_0|$, and

$$\partial_T f = \frac{1}{r}\left(\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta}+\frac{\partial f}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial \vartheta}\right)=\mathrm{i}\cdot\left(\frac{z-z_0}{|z-z_0|}\cdot f_z-\frac{\overline{z-z_0}}{|z-z_0|}\cdot f_{\bar z}\right), \tag{6.1.17}$$

where $\vartheta=\arg(z-z_0)$ because $J_f(z)=|f_z|^2-|f_{\bar z}|^2$.

An ACL homeomorphism $f: D\to\mathbb{C}$ is called a *strong ring solution* of the Beltrami equation (B) with a complex coefficient μ if f satisfies (B) a.e., $f^{-1}\in W^{1,2}_{\rm loc}$ and f is a ring Q-homeomorphism at every point $z_0\in\overline{D}$ with $Q(z)=Q_{z_0}(z)=K^T_\mu(z,z_0)$; (see (6.1.6) and (6.1.14)). In the next chapter we show that strong ring solutions exist for wide classes of the degenerate Beltrami equations.

Note that the existence theorems in the next chapter (see also Theorem 6.1 and Corollary 6.1 in the next section) show that $Q(z)$ for ring Q-homeomorphisms can be less than 1 on a set of positive measure in a neighborhood of a point z_0 because $K^T_\mu(z,z_0)$ is so. This is the essential difference of ring Q-homeomorphisms in comparison with Q-homeomorphisms; cf., e.g., [165].

6.2 Examples and Properties

The *big radial dilatation* of f at z with respect to z_0 is defined by

$$K^R(z,z_0,f)=\frac{|J_f(z)|}{|\partial^{z_0}_R f(z)|^2}, \tag{6.2.1}$$

where

$$|\partial^{z_0}_R f(z)| \;=\; \min_{\omega\in\mathbb{C},|\omega|=1}\frac{|\partial_\omega f(z)|}{|\mathrm{Re}\,\omega\overline{\omega_0}|}\,. \tag{6.2.2}$$

Here $\mathrm{Re}\,\omega\overline{\omega_0}$ is the scalar product of vectors ω and ω_0. In the other words, $\mathrm{Re}\,\omega\overline{\omega_0}$ is the projection of the vector ω onto the radial direction ω_0. Obviously, there is a unit vector ω_* such that

$$|\partial^{z_0}_R f(z)| \;=\; \frac{|\partial_{\omega_*} f(z)|}{|\mathrm{Re}\,\omega_*\overline{\omega_0}|}\,. \tag{6.2.3}$$

It is clear that

$$|\partial^{z_0}_r f(z)| \;\ge\; |\partial^{z_0}_R f(z)| \;\ge\; \min_{\omega\in\mathbb{C},|\omega|=1}|\partial_\omega f(z)|, \tag{6.2.4}$$

and hence,

$$K^r(z,z_0,f) \;\leq\; K^R(z,z_0,f) \;\leq\; K_\mu(z), \tag{6.2.5}$$

and the equalities hold in (6.2.5) if and only if the minimum in the right-hand side of (6.2.4) is realized for the radial direction $\omega = \omega_0$.

Note that $\partial_r^{z_0} f(z) \neq 0$, $|\partial_R^{z_0} f(z)| \neq 0$ and $\partial_T^{z_0} f(z) \neq 0$ at every regular point $z \neq z_0$ of f; see, e.g., 1.2.1 in [202]. In view of (6.1.11), (6.1.13), and (6.1.14), the following lemma shows that the big radial dilatation coincides with the tangential dilatation at every regular point:

Lemma 6.1. *Let $z \in D$ be a regular point of a mapping $f : D \to \mathbb{C}$ with the complex dilatation $\mu(z) = f_{\bar z}/f_z$ such that $|\mu(z)| < 1$. Then*

$$K^R(z,z_0,f) \;=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \qquad \forall\, z_0 \in \mathbb{C}. \tag{6.2.6}$$

Proof. The derivative of f at the regular point z in the arbitrary direction $\omega = \mathrm{e}^{\mathrm{i}\alpha}$ is the quantity $\partial_\omega f(z) = f_z + f_{\bar z}\cdot \mathrm{e}^{-2\mathrm{i}\alpha}$; see, e.g., [152], pp. 17 and 182. Consequently,

$$\begin{aligned} X \;:\; &= \frac{|\partial_R^{z_0} f(z)|^2}{|f_z|^2} \;=\; \min_{\alpha\in[0,2\pi]} \frac{|\mu(z)+\mathrm{e}^{2\mathrm{i}\alpha}|^2}{\cos^2(\alpha-\vartheta)} \;=\; \min_{\beta\in[0,2\pi]} \frac{|\nu-\mathrm{e}^{2\mathrm{i}\beta}|^2}{\sin^2\beta} \\ &= \min_{\beta\in[0,2\pi]} \frac{1+|\nu|^2-2|\nu|\cos(\kappa-2\beta)}{\sin^2\beta} \\ &= \min_{t\in[-1,1]} \frac{1+|\nu|^2-2|\nu|\cdot[(1-2t^2)\cos\kappa \pm 2t(1-t^2)^{1/2}\sin\kappa]}{t^2}, \end{aligned}$$

where $t = \sin\beta$, $\beta = \alpha + \frac{\pi}{2} - \vartheta$, $\nu = \mu(z)\mathrm{e}^{-2\mathrm{i}\vartheta}$ and $\kappa = \arg\nu = \arg\mu - 2\vartheta$. Hence

$$X \;=\; \min_{\tau\in[1,\infty]} \varphi_\pm(\tau) \quad \text{where} \quad \tau = 1/\sin^2\beta,\;\; \varphi_\pm(\tau) = b + a\tau \pm c(\tau-1)^{1/2},$$

$$a \;=\; 1 \,+\, |\nu|^2 \,-\, 2\,|\nu|\,\cos\kappa, \quad b \;=\; 4\,|\nu|\,\cos\kappa, \quad c \;=\; 4\,|\nu|\,\sin\kappa.$$

Since $\varphi_\pm'(\tau) = a \pm (\tau-1)^{-1/2}c/2$ the minimum is realized for $\tau = 1 + c^2/4a^2$ under $(\tau-1)^{1/2} = \mp c/2a$, correspondingly, where the signs are opposite. Thus,

$$X \;=\; b + \left(a + \frac{1}{4}\frac{c^2}{a}\right) - \frac{1}{2}\frac{c^2}{a} \;=\; \frac{(1-|\nu|^2)^2}{1+|\nu|^2-2|\nu|\cos\kappa},$$

that implies (6.2.6). □

Some prototypes of the following theorem can be found in [99, 147] and [198]. In these theorems, both $|\mu|$ and $\arg\mu$ are incorporated in modulus estimations.

Theorem 6.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{\rm loc}$ such that $f^{-1} \in W^{1,2}_{\rm loc}$. Then at every point $z_0 \in \overline{D}$, the mapping f is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z,z_0)$, where $\mu(z) = \mu_f(z)$.*

Proof. Fix $z_0 \in \overline{D}$, and r_1 and r_2 such that $0 < r_1 < r_2 < r_0 \le \sup_{z\in D} |z-z_0|$, and let $S_1 = \{z \in \mathbb{C} : |z-z_0| = r_1\}$ and $S_2 = \{z \in \mathbb{C} : |z-z_0| = r_2\}$. Set $\Gamma = \Delta(S_1,S_2,D)$ and denote by Γ_* the family of all rectifiable paths $\gamma_* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$ by the Fuglede theorem (see [85] and [258]) because $f^{-1} \in \mathrm{ACL}^2$; see, e.g., [171], p. 8.

Note that, for any continua C_0 and C_1 in D contained in the different connected components of the complement of the ring

$$A = A(z_0,r_1,r_2) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}$$

in $\mathbb{C}$, the family $\Delta(C_0,C_1,D)$ is minorized by the family $\Delta(S_1,S_2,A)$, i.e., every path in $\Delta(C_0,C_1,D)$ contains a path in $\Delta(S_1,S_2,A)$ as its subpath. Hence, $M(\Delta(C_0,C_1,D)) \le M(\Delta(S_1,S_2,A))$, and thus, it is sufficient to prove the inequality

$$M(\Delta(S_1,S_2,A)) \;\le\; \int\limits_A Q(z)\;\eta^2(|z-z_0|)\;\mathrm{d}m(z)$$

for every measurable function, $\eta : (r_1,r_2) \to [0,\infty]$ with $\int_{r_1}^{r_2} \eta(r)\mathrm{d}r = 1$.

Fix $\gamma_* \in \Gamma_*$. Set $\gamma = f^{-1}\circ\gamma_*$ and denote by s and s_* natural (length) parameters of γ and γ_*, respectively. Note that the correspondence $s_*(s)$ between the natural parameters of γ_* and γ is a strictly monotone function and we may assume that $s_*(s)$ is increasing. For $\gamma_* \in \Gamma_*$, the inverse function $s(s_*)$ has the (N) property and $s_*(s)$ is differentiable a.e. as a monotone function. Thus, $(\mathrm{d}s_*)/(\mathrm{d}s) \ne 0$ a.e. on γ by [189]. Let s be such that $z = \gamma(s)$ is a regular point for f and suppose that γ is differentiable at s with $(\mathrm{d}s_*)/(\mathrm{d}s) \ne 0$. Let $r = |z-z_0|$ and let ω be a unit tangential vector to the path γ at the point $z = \gamma(s)$. Then

$$\left|\frac{\mathrm{d}r}{\mathrm{d}s_*}\right| = \frac{\frac{\mathrm{d}r}{\mathrm{d}s}}{\frac{\mathrm{d}s_*}{\mathrm{d}s}} = \frac{|\mathrm{Re}\,\omega\overline{\omega_0}|}{|\partial_\omega f(z)|} \le \frac{1}{|\partial^{z_0}_R f(z)|}, \tag{6.2.7}$$

where $|\partial^{z_0}_R f(z)|$ is defined by (6.2.2).

Now, let $\eta : (r_1,r_2) \to [0,\infty]$ be an arbitrary measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r)\;\mathrm{d}r \;=\; 1. \tag{6.2.8}$$

By the Lusin theorem, there is a Borel function $\eta_* : (r_1,r_2) \to [0,\infty]$ such that $\eta_*(r) = \eta(r)$ a.e. (see, e.g., 2.3.5 in [83] and [Sa], p. 69). Let

$$\rho(z) = \eta_*(|z-z_0|)$$

in the ring $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $\rho(z) = 0$ outside of A. Set

$$\rho_*(w) = \{\rho / |\partial_R^{z_0} f|\} \circ f^{-1}(w)$$

if $z = f^{-1}(w)$ is a regular point of f, $\rho_*(w) = \infty$ at the rest points of $f(D)$ and $\rho_*(w) = 0$ outside $f(D)$. Then by (6.2.7) and (6.2.8), for $\gamma_* \in \Gamma_*$,

$$\int_{\gamma_*} \rho_* \mathrm{d}s_* \ \geq \ \int_{\gamma_*} \eta(r) \left| \frac{\mathrm{d}r}{\mathrm{d}s_*} \right| \mathrm{d}s_* \ \geq \ \int_{\gamma_*} \eta(r) \, \frac{\mathrm{d}r}{\mathrm{d}s_*} \, \mathrm{d}s_* \ = \ \int_{r_1}^{r_2} \eta(r) \, \mathrm{d}r \ = \ 1$$

because the function $z = \gamma(s(s_*))$ is absolutely continuous and hence so is $r = |z - z_0|$ as a function of the parameter s_*. Consequently, ρ_* is admissible for all $\gamma_* \in \Gamma_*$.

By Proposition 2.5, f and f^{-1} are regular a.e. and have the property (N). Thus, by change of variables (see, e.g., Theorem 6.4 in [177]), we have in view of Lemma 6.1 that

$$M(f\Gamma) \leq \int_{f(A)} \rho_*(w)^2 \mathrm{d}u\mathrm{d}v \ = \ \int_A \rho(z)^2 \, K_\mu^T(z, z_0) \, \mathrm{d}m(z)$$

$$= \int_A K_\mu^T(z, z_0) \cdot \eta^2(|z - z_0|) \, \mathrm{d}m(z),$$

i.e., f is a ring Q-homeomorphism with $Q(z) = K_\mu^T(z, z_0)$. □

If f is a plane $W^{1,2}_{\mathrm{loc}}$ homeomorphism with a locally integrable $K_f(z)$, then $f^{-1} \in W^{1,2}_{\mathrm{loc}}$; see, e.g., [104]. Hence, we obtain the following consequence of Theorem 6.1 which will be quoted further:

Corollary 6.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{\mathrm{loc}}$ and suppose that $K_f(z) \in L^1_{\mathrm{loc}}$. Then f is a ring Q-homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = K_\mu^T(z, z_0)$, where $\mu(z) = \mu_f(z)$.*

Since $K_\mu^T(z, z_0) \leq K_\mu(z)$, we have also the following corollary:

Corollary 6.2. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{\mathrm{loc}}$ with $K_f \in L^1_{\mathrm{loc}}$. Then f is a ring Q-homeomorphism with $Q(z) = K_f(z)$.*

6.3 The Completeness of Ring Homeomorphisms

The following convergence theorem is crucial. It will allow us to derive new theorems on existence of strong ring solutions for the Beltrami equation.

Theorem 6.2. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1,2,\ldots$, be a sequence of ring Q-homeomorphisms at a point $z_0 \in \overline{D}$. If f_n converges locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then f is also a ring Q-homeomorphism at z_0.*

Proof. Note first that every point $w_0 \in D' = fD$ belongs to $D'_n = f_nD$ for all $n \geq N$ together with $\overline{D(w_0,\varepsilon)}$, where $D(w_0,\varepsilon) = \{w \in \overline{\mathbb{C}} : s(w,w_0) < \varepsilon\}$ for some $\varepsilon > 0$. Indeed, set $\delta = \frac{1}{2}\ s(z_0,\partial D)$, where $z_0 = f^{-1}(w_0)$ and $\varepsilon_n = s(w_0,\partial f_nD(z_0,\delta))$. Note that the sets $f_nD(z_0,\delta)$ are open and $\varepsilon_n > 0$ is the radius of the maximal closed disk centered at w_0 which is inside $\overline{f_nD(z_0,\delta)}$. Assume that $\varepsilon_n \to 0$ as $n \to \infty$. Since $\partial D(z_0,\delta)$ and $\partial f_nD(z_0,\delta) = f_n\partial D(z_0,\delta)$ are compact, there exist $z_n \in \partial D(z_0,\delta)$, $s(z_n,z_0) = \delta$, such that $\varepsilon_n = s(w_0, f_n(z_n))$, and we may assume that $z_n \to z_* \in \partial D(z_0,\delta)$ as $n \to \infty$, and then $f_n(z_n) \to f(z_*)$ as $n \to \infty$; see, e.g., [74], p. 268. However, by the construction, $s(w_0, f_n(z_n)) = \varepsilon_n \to 0$ as $n \to \infty$ and hence $f(z_*) = f(z_0)$, i.e., $z = z_*$. This contradiction disproves the above assumption. Thus, we obtain also that every compact set $C \subset D'$ belongs to D'_n for all $n \geq N$ for some N.

Now, we note that $D' = \bigcup_{m=1}^{\infty} C_m$ where $C_m = \overline{D^*_m}$, and D^*_m is a connected component of the open set $\Omega_m = \{w \in D' : s(w,\partial D') > 1/m\}$, $m = 1,2,\ldots$, including a fixed point $w_0 \in D'$. Indeed, every point $w \in D'$ can be joined with w_0 by a path γ in D'. Because $|\gamma|$ is compact, we have that $s(|\gamma|,\partial D') > 0$ and, consequently, $w \in D^*_m$ for large enough $m = 1,2,\ldots$.

Next, take an arbitrary pair of continua E and F in D which belong to the different connected components of the complement of a ring $A = A(z_0,r_1,r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, $z_0 \in \overline{D}$, $0 < r_1 < r_2 < r_0 \leq \sup_{z\in D} |z - z_0|$. For $m \geq m_0$, continua fE and fF belong to D^*_m. Fix one such m. Then the continua f_nE and f_nF also belong to D^*_m for large enough n. It is well known that

$$M(\Delta(f_nE, f_nF; D^*_m)) \ \to\ M(\Delta(fE, fF; D^*_m))$$

as $n \to \infty$; see Theorem 1 in [251]. However, $D^*_m \subset f_nD$ for large enough n and hence

$$M(\Delta(f_nE, f_nF; D^*_m)) \ \leq\ M(\Delta(f_nE, f_nF; f_nD))$$

and, thus, by definition of ring Q-homeomorphisms (see (6.1.6))

$$M(\Delta(fE, fF; D^*_m)) \ \leq \int\limits_{A\cap D} Q(z)\cdot\eta^2(|z - z_0|)\ \mathrm{d}m(z)$$

for every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ \mathrm{d}r \ =\ 1.$$

Finally, since $\Gamma = \bigcup_{m=m_0}^{\infty} \Gamma_m$ where $\Gamma = \Delta(fE, fF; fD)$ and $\Gamma_m :\, = \Delta(fE, fF; D^*_m)$ is increasing in $m = 1,2,\ldots$, we obtain that $M(\Gamma) = \lim_{m\to\infty} M(\Gamma_m)$ (see [272]), and thus

$$M(\Delta(fE,fF;fD)) \leq \int\limits_{A\cap D} Q(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z),$$

i.e., f is a ring Q-homeomorphism at z_0. □

6.4 One Integral Inequality

The following statement is a partial case of Lemma 3.1 for $n=2$ in [209]; cf. also a simpler version in Lemma 2.19.

Lemma 6.2. *Let $Q:\mathbb{D}\to[0,\infty]$ be a measurable function, and let $\Phi:[0,\infty]\to[0,\infty]$ be a nondecreasing convex function. Then*

$$\int\limits_{\varepsilon}^{1}\frac{\mathrm{d}r}{rq(r)} \geq \frac{1}{2}\int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^2}}\frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)} \qquad \forall\varepsilon\in(0,1), \tag{6.4.1}$$

where $q(r)$ is the average of the function Q over the circle $|z|=r$ and $M(\varepsilon)$ is the average of $\Phi\circ Q$ over the ring $\mathbb{C}(\varepsilon)=\{\,z\in\mathbb{C}:\ \varepsilon<|z|\ <\ 1\,\}$.

Proof. The result is obvious if $M(\varepsilon)=\infty$ because then the integral in the right-hand side in (6.4.1) is equal to zero. Hence, we may assume $M(\varepsilon)<\infty$. Moreover, by the same reasons, we may also assume that $M(\varepsilon)>0$. Denote

$$t_* = \sup_{\Phi(t)=\tau_0} t, \qquad \tau_0=\Phi(0). \tag{6.4.2}$$

Setting

$$H(t)\ :=\ \log\ \Phi(t), \tag{6.4.3}$$

we see that

$$H^{-1}(\eta)\ =\ \Phi^{-1}(\mathrm{e}^{\eta}),\quad \Phi^{-1}(\tau)\ =\ H^{-1}(\log\ \tau). \tag{6.4.4}$$

Thus, we obtain that

$$q(r)=H^{-1}\left(\log\frac{h(r)}{r^2}\right)=H^{-1}\left(2\log\frac{1}{r}+\log\ h(r)\right) \qquad \forall\ r\ \in R_*, \tag{6.4.5}$$

where $h(r):\ =\ r^2\Phi\,(q(r))$ and $R_*=\{\ r\in(\varepsilon,1):\ q(r)\ >\ t_*\}$. Then also

$$q(\mathrm{e}^{-s})\ =\ H^{-1}\left(2s\ +\ \log\ h(\mathrm{e}^{-s})\right) \qquad \forall\ s\ \in S_*, \tag{6.4.6}$$

where $S_*\ =\ \{s\in(0,\log\frac{1}{\varepsilon}):\ q\,(\mathrm{e}^{-s})\ >\ t_*\}$.

Now, by the Jensen inequality and convexity of Φ, we have that

$$\int_0^{\log\frac{1}{\varepsilon}} h(\mathrm{e}^{-s})\,\mathrm{d}s \;=\; \int_{\varepsilon}^{1} h(r)\,\frac{\mathrm{d}r}{r} \;=\; \int_{\varepsilon}^{1} \Phi(q(r))\,r\mathrm{d}r$$
$$\le \int_{\varepsilon}^{1}\left(\fint_{S(r)} \Phi(Q(x))\,\mathrm{d}\mathscr{A}\right) r\mathrm{d}r \;\le\; \frac{1}{2}\,M(\varepsilon), \qquad (6.4.7)$$

where we use the mean value of the function $\Phi\circ Q$ over the circle $C(r)=\{z\in\mathbb{C}:|z|=r\}$. Then arguing by contradiction, it is easy to see that

$$|T| \;=\; \int_T \mathrm{d}s \;\le\; \frac{1}{2}, \qquad (6.4.8)$$

where $T \;=\; \{\, s\in(0,\log\frac{1}{\varepsilon}):\quad h(\mathrm{e}^{-s})\; > M(\varepsilon)\}$. Next, let us show that

$$q\left(\mathrm{e}^{-s}\right) \;\le\; H^{-1}\left(2s \,+\, \log\, M(\varepsilon)\right) \qquad \forall\, s\in\left(0,\log\frac{1}{\varepsilon}\right)\setminus T_*, \qquad (6.4.9)$$

where $T_* \;=\; T\cap S_*$. Note that $\left(0,\log\frac{1}{\varepsilon}\right)\setminus T_* = \left[\left(0,\log\frac{1}{\varepsilon}\right)\setminus S_*\right]\cup\left[\left(0,\log\frac{1}{\varepsilon}\right)\setminus T\right] = \left[\left(0,\log\frac{1}{\varepsilon}\right)\setminus S_*\right]\cup[S_*\setminus T]$. The inequality (6.4.9) holds for $s\in S_*\setminus T$ by (6.4.6) because H^{-1} is a nondecreasing function. Note also that by (6.4.2),

$$\mathrm{e}^{2s}M(\varepsilon) > \Phi(0) = \tau_0 \qquad \forall\, s\in\left(0,\,\log\frac{1}{\varepsilon}\right), \qquad (6.4.10)$$

and then by (6.4.4),

$$t_* < \Phi^{-1}\left(\mathrm{e}^{2s}M(\varepsilon)\right) = H^{-1}\left(2s \,+\, \log\, M(\varepsilon)\right) \qquad \forall\, s\in\left(0,\,\log\frac{1}{\varepsilon}\right). \qquad (6.4.11)$$

Consequently, (6.4.9) holds for $s\in(0,\log\frac{1}{\varepsilon})\setminus S_*$, too. Thus, (6.4.9) is true.

Since H^{-1} is non-decreasing, we have by (6.4.8) and (6.4.9) that

$$\int_{\varepsilon}^{1}\frac{\mathrm{d}r}{rq(r)} \;=\; \int_0^{\log\frac{1}{\varepsilon}}\frac{\mathrm{d}s}{q(\mathrm{e}^{-s})} \;\ge\; \int_{\left(0,\log\frac{1}{\varepsilon}\right)\setminus T_*}\frac{\mathrm{d}s}{H^{-1}(2s+\Delta)}$$
$$\ge \int_{|T_*|}^{\log\frac{1}{\varepsilon}}\frac{\mathrm{d}s}{H^{-1}(ns+\Delta)} \;\ge\; \int_{\frac{1}{2}}^{\log\frac{1}{\varepsilon}}\frac{\mathrm{d}s}{H^{-1}(2s+\Delta)} \;=\; \frac{1}{2}\int_{1+\Delta}^{\log\frac{1}{\varepsilon}}\frac{\mathrm{d}\eta}{H^{-1}(\eta)}, \qquad (6.4.12)$$

where $\Delta = \log M(\varepsilon)$. Note that $1+\Delta = \log\, eM(\varepsilon)$. Thus,

$$\int\limits_{\varepsilon}^{1} \frac{\mathrm{d}r}{rq(r)} \geq \frac{1}{2} \int\limits_{\log eM(\varepsilon)}^{\log\frac{1}{\varepsilon}} \frac{\mathrm{d}\eta}{H^{-1}(\eta)} \tag{6.4.13}$$

and, after the substitution $\eta = \log\,\tau$, we obtain (6.4.1). □

6.5 Distortion Estimates

The following simple remark makes it possible to obtain a number of significant conclusions on ring Q-homeomorphisms:

Proposition 6.1. *If f is a ring Q-homeomorphism at a point $z_0 \in D$ for a measurable function $Q : B(z_0, r_0) \to [0,\infty]$ with $r_0 = \operatorname{dist}(z_0, \partial D)$, $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$, $C_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$, $0 < \varepsilon < \varepsilon_0 < r_0$, $A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$, then*

$$M(\Delta(fC_\varepsilon, fC_0)) \leq \omega(\varepsilon), \tag{6.5.1}$$

where

$$\omega(\varepsilon) = \frac{1}{I^2(\varepsilon)} \int\limits_{A(\varepsilon,\varepsilon_0)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|)\, \mathrm{d}m(z) \tag{6.5.2}$$

for every one-parameter family of measurable functions $\psi_\varepsilon : [0,\infty] \to [0,\infty]$, $0 < \varepsilon < \varepsilon_0 < r_0$, such that

$$0 < I(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\, \mathrm{d}t < \infty \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{6.5.3}$$

Indeed, (6.5.1) follows with choosing $\eta(r) = \psi_\varepsilon(r)/I(\varepsilon)$, $r \in (\varepsilon, \varepsilon_0)$, in (6.1.6).

Using Proposition 6.1, we present now a sharp capacity estimate for ring Q-homeomorphisms. This estimate depends only on Q, and it implies as a special case a known inequality, which was proved for quasiconformal mappings by Reich and Walczak in [198] and that has later on been applied by several authors.

Below, we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$; see, e.g., [224], p. 6.

Lemma 6.3. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : B(z_0, r_0) \to [0,\infty]$ a measurable function and $q(r)$ the mean of $Q(z)$ over the circle $|z - z_0| = r$. For $0 < r_1 < r_2 < r_0$, set*

$$I = I(r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{rq(r)} \tag{6.5.4}$$

and $C_j = \{z \in \mathbb{C} : |z - z_0| = r_j\}$, $j = 1,2$. *Then*

$$M(\Delta(fC_1, fC_2)) \leq \frac{2\pi}{I} \tag{6.5.5}$$

whenever f is a ring Q-homeomorphism of $D_0 = D \setminus \{z_0\}$ into $\mathbb{C}$ at z_0.

Proof. With no loss of generality, we may assume that $I \neq 0$, because otherwise (6.5.5) is trivial, and that $I \neq \infty$, because otherwise we can replace $Q(z)$ by $Q(z) + \delta$ with arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (6.5.5). The condition $I \neq \infty$ implies, in particular, that $q(r) \neq 0$ a.e. in (r_1, r_2).

For $I \neq 0, \infty$, Lemma 6.3 follows from Proposition 6.1 by choosing

$$\psi_\varepsilon(t) \equiv \psi(t) := \begin{cases} 1/[tq(t)], & t \in (0, \varepsilon_0), \\ 0, & \text{otherwise}, \end{cases} \tag{6.5.6}$$

with $\varepsilon = r_1$ and $\varepsilon_0 = r_2$ because

$$\int_A Q(z) \cdot \psi^2(|z - z_0|)\, \mathrm{d}m(z) = 2\pi I, \tag{6.5.7}$$

where $A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$. □

Corollary 6.3. *For every ring Q-homeomorphism f of $D_0 = D \setminus \{z_0\}$ into $\mathbb{C}$ at $z_0 \in D$ and $0 < r_1 < r_2 < r_0 = \operatorname{dist}(z_0, \partial D)$,*

$$\int_{r_1}^{r_2} \frac{\mathrm{d}r}{rq(r)} < \infty, \tag{6.5.8}$$

where $q(r)$ is the mean of $Q(z)$ over the circle $|z - z_0| = r$.

Indeed, by (2.5.2), (2.5.3), and (2.5.15) with $E = fC_1$ and $F = fC_2$,

$$M(\Delta(fC_1, fC_2)) \geq \frac{2\pi}{\log \frac{32}{\delta(fC_1)\delta(fC_2)}}. \tag{6.5.9}$$

The right-hand side in (6.5.9) is positive because f is injective. Thus, Corollary 6.3 follows from (6.5.5) in Lemma 6.3.

Corollary 6.4. *Let $f : D \to \mathbb{C}$ be a $W^{1,2}_{\mathrm{loc}}$ homeomorphism in a domain $D \subset \mathbb{C}$ such that*

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \in L^1_{\mathrm{loc}}(D), \tag{6.5.10}$$

where $\mu(z) = \mu_f(z)$*. Set*

$$q^T_{z_0}(r) \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{|1-\mathrm{e}^{-2\mathrm{i}\vartheta}\mu(z_0+r\mathrm{e}^{\mathrm{i}\vartheta})|^2}{1-|\mu(z_0+r\mathrm{e}^{\mathrm{i}\vartheta})|^2}\;\mathrm{d}\vartheta. \tag{6.5.11}$$

Then

$$\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{rq^T_{z_0}(r)} \;<\; \infty \tag{6.5.12}$$

for every $z_0 \in D$ *and* $0 < r_1 < r_2 < d_0$ *where* $d_0 = \operatorname{dist}(z_0, \partial D)$.

Corollary 6.4 follows from Corollaries 6.1 and 6.3 and from the definition of the tangential dilatation $K^T_\mu(z,z_0)$.

Corollary 6.5. *Let* $f : D \to \mathbb{C}$ *be a* $W^{1,2}_{\mathrm{loc}}$ *homeomorphism with* $K_\mu(z) \in L^1_{\mathrm{loc}}$, *where* $\mu(z) = \mu_f(z)$*. Then*

$$M(\Delta(fC_1, fC_2)) \;\le\; \left[\,\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{r\int\limits_0^{2\pi} \frac{|1-\mathrm{e}^{-2\mathrm{i}\vartheta}\mu(z_0+r\mathrm{e}^{\mathrm{i}\vartheta})|^2}{1-|\mu(z_0+r\mathrm{e}^{\mathrm{i}\vartheta})|^2}\,\mathrm{d}\vartheta}\right]^{-1}. \tag{6.5.13}$$

Indeed, by Corollary 6.1, f is a ring Q-homeomorphism at z_0 with $Q(z) = K^T_\mu(z,z_0)$. Thus, (6.5.13) follows from Lemma 6.3.

Remark 6.1. Equation (6.5.13) was first derived by Reich and Walczak [198] for quasiconformal mappings, and then by Lehto [147] for certain μ-homeomorphisms, and was later applied in [56, 57] and [98, 99] to the study of degenerate Beltrami equations.

The following lemma shows that the estimate (6.5.5), which implies (6.5.13), cannot be improved in the class of all ring Q-homeomorphisms. Note that the additional condition (6.5.14) which appears in the following lemma holds automatically for every ring Q-homeomorphism by Corollary 6.3:

Lemma 6.4. *Fix* $0 < r_1 < r_2 < r_0$, $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, *and suppose that* $Q : B(z_0, r_0) \to [0, \infty]$ *is a measurable function such that*

$$c_0 \;:=\; \int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{rq(r)} \;<\; \infty, \tag{6.5.14}$$

where $q(r)$ *is the average of* $Q(z)$ *over the circle* $|z - z_0| = r$, *and let*

$$\eta_0(r) = \frac{1}{c_0 rq(r)}\,. \tag{6.5.15}$$

Then

$$\frac{2\pi}{c_0} = \int\limits_A Q(z)\cdot\eta_0^2(|z-z_0|)\,\mathrm{d}m(z) \le \int\limits_A Q(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z) \tag{6.5.16}$$

for any $\eta : (r_1,r_2)\to[0,\infty]$ *such that*

$$\int\limits_{r_1}^{r_2}\eta(r)\,\mathrm{d}r = 1. \tag{6.5.17}$$

Proof. If $c_0=0$, then $q(r)=\infty$ for a.e. $r\in(r_1,r_2)$ and both sides in (6.5.16) are equal to ∞. Hence, we may assume below that $0<c_0<\infty$.

Now, by (6.5.14) and (6.5.17), $q(r)\neq 0$ and $\eta(r)\neq\infty$ a.e. in (r_1,r_2). Set $\alpha(r)=rq(r)\eta(r)$ and $w(r)=1/rq(r)$. Then by the standard conventions, $\eta(r)=\alpha(r)w(r)$ a.e. in (r_1,r_2) and

$$C := \int\limits_A Q(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z) = 2\pi\int\limits_{r_1}^{r_2}\alpha^2(r)\cdot w(r)\,\mathrm{d}r. \tag{6.5.18}$$

By Jensen's inequality with weights (see, e.g., Theorem 2.6.2 in [196]), applied to the convex function $\varphi(t)=t^2$ in the interval $\Omega=(r_1,r_2)$ with the probability measure

$$\nu(E) = \frac{1}{c_0}\int\limits_E w(r)\,\mathrm{d}r, \tag{6.5.19}$$

we obtain that

$$\left(⨍\alpha^2(r)w(r)\,\mathrm{d}r\right)^{1/2} \ge ⨍\alpha(r)w(r)\,\mathrm{d}r = \frac{1}{c_0}, \tag{6.5.20}$$

where we also used the fact that $\eta(r)=\alpha(r)w(r)$ satisfies (6.5.17). Thus,

$$C \ge \frac{2\pi}{c_0} \tag{6.5.21}$$

and the proof is complete. □

Given a number $\Delta\in(0,1)$, a domain $D\subset\mathbb{C}$, a point $z_0\in D$, a number $r_0\le \operatorname{dist}(z_0,\partial D)$, and a measurable function $Q: B(z_0,r_0)\to[0,\infty]$, let $\mathfrak{R}_Q^\Delta$ denote the class of all ring Q-homeomorphisms f of $D_0=D\setminus\{z_0\}$ into $\overline{\mathbb{C}}$ at z_0 such that

$$\delta((\overline{\mathbb{C}}\setminus fD_0)\setminus E_0)\ge\Delta, \tag{6.5.22}$$

where E_0 denotes the connected component of $\overline{\mathbb{C}} \setminus fD_0$ including the cluster set $C(z_0, f)$. Next, we introduce the classes $\mathfrak{B}_Q^{\Delta}$ and $\mathfrak{F}_Q^{\Delta}$ of certain quasiconformal mappings. Let $\mathfrak{B}_Q^{\Delta}$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (6.5.22) such that

$$K_\mu^T(z, z_0) \;=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \;\leq\; Q(z) \quad \text{a.e. in } B(z_0, r_0), \tag{6.5.23}$$

where $\mu = \mu_f$. Similarly, let $\mathfrak{F}_Q^{\Delta}$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (6.5.22) such that

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \;\leq\; Q(z) \quad \text{a.e. in } B(z_0, r_0). \tag{6.5.24}$$

Remark 6.2. By Corollaries 6.1 and the relations (6.2.5) and (6.1.13),

$$\mathfrak{F}_Q^{\Delta} \subset \mathfrak{B}_Q^{\Delta} \subset \mathfrak{R}_Q^{\Delta}\,. \tag{6.5.25}$$

Combining Lemma 2.13 and Proposition 6.1, we obtain the following general distortion estimate in the class $\mathfrak{R}_Q^{\Delta}$:

Corollary 6.6. *Let $f \in \mathfrak{R}_Q^{\Delta}$, and let $\omega(\varepsilon)$ be as in Proposition 6.1. Then*

$$s(f(\zeta), f(\zeta')) \;\leq\; \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{\omega(|\zeta - z_0|)}\right) \tag{6.5.26}$$

for all $\zeta \in B(z_0, \varepsilon_0) \setminus \{z_0\}$ and $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$.

In particular, choosing $\psi_\varepsilon(t) \equiv \psi(t)$, $\varepsilon \in (0, \varepsilon_0)$ in Proposition 6.1, we obtain the following statement:

Theorem 6.3. *Let $f \in \mathfrak{R}_Q^{\Delta}$, and let $\psi : [0, \infty] \to [0, \infty]$ be a measurable function such that*

$$0 \;<\; \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, \mathrm{d}t \;<\; \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{6.5.27}$$

Suppose that

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\;\, \mathrm{d}m(z) \;\leq\; C \cdot \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, \mathrm{d}t \tag{6.5.28}$$

for all $\varepsilon \in (0, \varepsilon_0)$. *Then*

$$s(f(\zeta), f(\zeta')) \leq \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{C} \cdot \int\limits_{|\zeta - z_0|}^{\varepsilon_0} \psi(t)\, \mathrm{d}t \right) \tag{6.5.29}$$

whenever $f \in \mathfrak{R}_Q^\Delta$ *and* $\zeta \in B(z_0, \varepsilon_0) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$.

Next, choosing in Theorem 6.3 the special functional parameter $\psi(t)$ given by (6.5.6), we obtain the following distortion theorem for ring Q-homeomorphisms:

Theorem 6.4. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : B(z_0, r_0) \to [0, \infty]$ a measurable function and let $f \in \mathfrak{R}_Q^\Delta$. Then*

$$s(f(\zeta), f(\zeta')) \leq \frac{32}{\Delta} \cdot \exp\left(- \int\limits_{|\zeta - z_0|}^{r_0} \frac{\mathrm{d}r}{rq(r)} \right) \tag{6.5.30}$$

for all $\zeta \in B(z_0, r_0) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$, *where* $q(r)$ *is the mean value of* $Q(z)$ *over the circle* $|z - z_0| = r$.

Combining Theorem 6.4 and Lemma 6.2, we obtain also the following:

Theorem 6.5. *Let D be a domain in $\mathbb{C}$, z_0 a point in D, $r_0 \leq \operatorname{dist}(z_0, \partial D)$, $Q : B(z_0, r_0) \to [0, \infty]$ a measurable function and let $f \in \mathfrak{R}_Q^\Delta$. Then for every increasing convex function $\Phi : [0, \infty] \to [0, \infty]$,*

$$s(f(\zeta), f(\zeta')) \leq \frac{32}{\Delta} \cdot \exp\left(-\frac{1}{2} \int\limits_{eM}^{\frac{r_0}{|\zeta - z_0|}} \frac{\mathrm{d}\tau}{\tau \Phi^{-1}(\tau)} \right) \tag{6.5.31}$$

for all $\zeta \in B(z_0, r_0) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$ *where M is the mean value of $\Phi \circ Q$ over the ring* $R = \{z \in \mathbb{C} : |\zeta - z_0| < |z - z_0| < r_0\}$.

Remark 6.3. Of course, the estimate (6.5.31) is effective if M is bounded, and either the condition $\int_{\delta_*}^{\infty} \mathrm{d}\tau / \tau \Phi^{-1}(\tau) = \infty$, $\delta_* > \Phi(0)$, or one of the equivalent conditions (2.8.12)–(2.8.16) holds.

In the following theorem, the estimate of distortion is expressed in terms of maximal dispersion; see (2.3.5).

Theorem 6.6. *Let $f \in \mathfrak{R}_Q^\Delta$ for $\Delta > 0$ and Q with finite mean oscillation at $z_0 \in D$. If Q is integrable over a disk $B(z_0, \varepsilon_0) \subset D$, then*

$$s(f(\zeta), f(\zeta')) \leq \frac{32}{\Delta} \cdot \left(\log \frac{2\varepsilon_0}{|\zeta - z_0|} \right)^{-\beta_0} \tag{6.5.32}$$

for all $\zeta \in B(z_0, \varepsilon_0/2) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$ *where*

$$\beta_0 = (2q_0 + 3\mathrm{e}^2 d_0)^{-1}, \tag{6.5.33}$$

q_0 *is the mean and* d_0 *the maximal dispersion of* $Q(z)$ *in* $B(z_0, \varepsilon_0)$.

Proof. The mean and the dispersion of a function over disks are invariant under linear transformations $w = (z - z_0)/2\varepsilon_0$. Hence, (6.5.32) follows by Theorem 6.3 and Lemma 5.3. □

The following two corollaries, which are formulated in terms of the mean of Q over disks and annuli, are obtained from Corollary 6.6 by setting $\psi_\varepsilon(t) \equiv 1$ for $0 < \varepsilon < \varepsilon_0$ in (6.5.2), where we choose $\varepsilon = |\zeta - z_0|$ and $\varepsilon_0 = 4|\zeta - z_0|$ in the case of Corollary 6.7 and $\varepsilon = |\zeta - z_0|$ and $\varepsilon_0 = 3|\zeta - z_0|$ in the case of Corollary 6.8:

Corollary 6.7. *Let* $Q : B(z_0, r_0) \to [0, \infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$, *be a measurable function, for* $r \le r_0/4$, *and let* $M_Q(r)$ *denote the mean of* Q *over the disk* $B(z_0, 4r)$, *and let* $\Delta > 0$. *If* $f \in \mathfrak{R}_Q^\Delta$, *then*

$$s(f(\zeta), f(\zeta')) \le \frac{32}{\Delta} \cdot \mathrm{e}^{-1/M_Q(|\zeta - z_0|)} \tag{6.5.34}$$

for all $\zeta \in B(z_0, r_0/4) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$.

Corollary 6.8. *Let* $Q : B(z_0, r_0) \to [0, \infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$, *be a measurable function, for* $r \le r_0/3$, *let* $M^Q(r)$ *denote the mean of* Q *over the ring* $A = \{z \in \mathbb{C} : r < |z - z_0| < 3r\}$, *and let* $\Delta > 0$. *If* $f \in \mathfrak{R}_Q^\Delta$, *then*

$$s(f(\zeta), f(\zeta')) \le \frac{32}{\Delta} \cdot \mathrm{e}^{-1/M^Q(|\zeta - z_0|)} \tag{6.5.35}$$

for all $\zeta \in B(z_0, r_0/3) \setminus \{z_0\}$ *and* $\zeta' \in B(z_0, |\zeta - z_0|) \setminus \{z_0\}$.

Another consequence of Lemma 2.13 and Proposition 6.1 (see Corollary 6.6 above) can be formulated in terms of the *logarithmic mean* of Q over the ring $A(\varepsilon) = A(z_0, \varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$ which is defined by

$$M_{\log}^Q(\varepsilon) = \fint_{\varepsilon}^{\varepsilon_0} q(t)\, \mathrm{d}\log t \ := \ \frac{1}{\log \varepsilon_0/\varepsilon} \int_{\varepsilon}^{\varepsilon_0} q(t)\, \frac{\mathrm{d}t}{t}, \tag{6.5.36}$$

where $q(t)$ denotes the mean value of Q over the circle $|z - z_0| = t$. Choosing in the expression (6.5.2) $\psi_\varepsilon(t) = 1/t$ for $0 < \varepsilon < \varepsilon_0$, and setting $\varepsilon = |\zeta - z_0|$, we have the following statement:

Corollary 6.9. *Let* $Q : B(z_0, r_0) \to [0, \infty]$, $r_0 \le \operatorname{dist}(z_0, \partial D)$, *be a measurable function,* $\varepsilon_0 \in (0, r_0)$ *and* $\Delta > 0$. *If* $f \in \mathfrak{R}_Q^\Delta$, *then*

$$s(f(\zeta),f(\zeta')) \leq \frac{32}{\Delta}\cdot\left(\frac{|\zeta-z_0|}{\varepsilon_0}\right)^{1/M^Q_{\log}(|\zeta-z_0|)} \tag{6.5.37}$$

for all $\zeta\in B(z_0,\varepsilon_0)\setminus\{z_0\}$ *and* $\zeta'\in B(z_0,|\zeta-z_0|)\setminus\{z_0\}$.

Note that, for $Q\equiv K\in[1,\infty)$, (6.5.37) implies the following known distortion estimate for quasiconformal mappings:

$$s(f(\zeta),f(z_0)) \leq \frac{32}{\Delta}\cdot\left(\frac{|\zeta-z_0|}{\varepsilon_0}\right)^{1/K}. \tag{6.5.38}$$

The corollaries and theorems presented above show that Lemma 2.13 and Proposition 6.1 are useful tools in deriving various distortion estimates for ring Q-homeomorphisms. These, in turn, are useful in the study of properties of ring Q-homeomorphisms and, in particular, of strong ring solutions of the Beltrami equation (B), where $Q(z)$ can be either the maximal dilatation $K_\mu(z)$ or the tangential dilatation $K^T_\mu(z,z_0)$ that are defined in (6.1.1) and (6.1.11), respectively.

6.6 On Removability of Isolated Singularities

It is well known that isolated singularities are removable for conformal as well as for quasiconformal mappings. The following statement shows that any power of integrability of $Q(z)$ cannot guarantee the removability of isolated singularities of ring Q-homeomorphisms. This is a new phenomenon.

Proposition 6.2. *For any* $p\in[1,\infty)$*, there is a ring* Q*-homeomorphism* f *of* $\mathbb{D}\setminus\{0\}$ *into* $\mathbb{C}$ *at* 0 *with* $Q\in L^p(\mathbb{D})$ *which has no continuous extension to* $\mathbb{D}$*. Furthermore, a locally quasiconformal mapping in* $\mathbb{D}\setminus\{0\}$ *can be chosen as* f*.*

Here, $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denotes the unit disk in $\mathbb{C}$.

Proof. The desired homeomorphism f can be given in the explicit form

$$\zeta = f(z) = \frac{z}{|z|}\,(1+|z|^\alpha),$$

where $\alpha\in(0,2/p)$. Note that $f\in C^\infty$ and maps the punctured unit disk $\mathbb{D}\setminus\{0\}$ onto the ring $1<|\zeta|<2$ in $\mathbb{C}$ and f has no continuous extension onto $\mathbb{D}$.

On the circle $|z|=r$, the tangent and radial distortion are

$$\delta_\tau = \frac{|\zeta|}{|z|} = \frac{1+r^\alpha}{r},\quad \delta_r = \frac{\partial|\zeta|}{\partial|z|} = \alpha r^{\alpha-1},$$

respectively. Without loss of generality, we may assume that p is great enough so that $\alpha < 1$. Thus, $\delta_\tau > \delta_r$, and by the circle symmetry, the maximal dilatation of the mapping f at every point $z \in \mathbb{D} \setminus \{0\}$ is

$$K_f(z) = \frac{\delta_\tau}{\delta_r} = \frac{1+r^\alpha}{\alpha r^\alpha} \leq \frac{C}{r^\alpha}, \quad |z| = r,$$

where $C = 2/\alpha$. Note that $K_f \in L^p(\mathbb{D})$ because $\alpha p < 2$ by the choice of α. It remains to note that the mapping f is locally quasiconformal in $\mathbb{D}\setminus\{0\}$ and hence f is a Q-homeomorphism with $Q(z) = K_f(z)$; see Theorem 6.1 and Corollary 6.2. □

However, it suffices for removability of isolated singularities of ring Q-homeomorphisms to require the integrability of $Q(z)$ with suitable weights; see the next general lemma, which follows immediately by Theorem 6.3.

Lemma 6.5. *Let $f : \mathbb{D}\setminus\{0\} \to \mathbb{C}$ be a ring Q-homeomorphism at 0. If*

$$\int\limits_{\varepsilon<|z|<1} Q(z) \cdot \psi^2(|z|) \;\, \mathrm{d}m(z) = o(I(\varepsilon)^2) \tag{6.6.1}$$

as $\varepsilon \to 0$, where $\psi(t)$ is a nonnegative measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon) = \int\limits_{\varepsilon}^{1} \psi(t) \;\, \mathrm{d}t < \infty \qquad \forall\, \varepsilon \in (0,1), \tag{6.6.2}$$

then f has a continuous extension to $\mathbb{D}$.

Remark 6.4. Note that the conditions (6.6.1)–(6.6.2) imply $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately with arguments by contradiction. Note also that (6.6.1) holds, in particular, if

$$\int\limits_{\mathbb{D}} Q(z) \cdot \psi^2(|z|) \;\, \mathrm{d}m(z) < \infty \tag{6.6.3}$$

and $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the removability of a singularity of the ring Q-homeomorphism f at $z = 0$, it is sufficient that the integral (6.6.3) converges for some nonnegative function $\psi(t)$ which is locally integrable over $(0,1)$ but has a nonintegrable singularity at 0. The functions $Q(z) = \log^\lambda(e/|z|)$, $\lambda \in (0,1)$, $z \in \mathbb{D} \setminus \{0\}$, and $\psi(t) = 1/(t\log(e/t))$, $t \in (0,1)$, show that (6.6.3) is compatible with the condition $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

In particular, choosing $\psi(t) = 1/(t\log 1/t)$ in Lemma 6.5, we obtain by Lemma 5.3 the following theorem:

Theorem 6.7. *Let D be a domain in $\mathbb{C}$, $z_0 \in D$, and let $f : D\setminus\{z_0\} \to \mathbb{C}$ be a ring Q-homeomorphism at z_0 where $Q(z)$ has finite mean oscillation at z_0. Then f can be extended to D by continuity in $\overline{\mathbb{C}}$.*

In other words, an isolated singularity of a ring Q-homeomorphism is removable if $Q(z)$ has finite mean oscillation at the point. In particular, this is the case if $Q(z)$ is continuous at z_0. As consequences of Theorem 6.7, Proposition 2.2, and Corollary 2.1, we also obtain the following statements:

Corollary 6.10. *A Lebesgue point of Q is a removable isolated singularity for ring Q-homeomorphisms.*

Corollary 6.11. *If $f : D\setminus\{z_0\} \to \mathbb{C}$ is a ring Q-homeomorphism at $z_0 \in D$ with*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} Q(z)\, \mathrm{d}m(z) \ < \ \infty, \tag{6.6.4}$$

then f can be extended to D by continuity in $\overline{\mathbb{C}}$.

Similarly, choosing in Lemma 6.5 the function $\psi(t) = 1/t$ as a weight, we come to the following statement:

Theorem 6.8. *Let D be a domain in $\mathbb{C}$, $z_0 \in D$, and let $f : D\setminus\{z_0\} \to \mathbb{C}$ be a ring Q-homeomorphism at z_0. If*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\ \frac{\mathrm{d}m(z)}{|z-z_0|^2} = o\left(\left[\log\frac{\varepsilon_0}{\varepsilon}\right]^2\right) \tag{6.6.5}$$

as $\varepsilon \to 0$, then f can be extended to D by continuity in $\overline{\mathbb{C}}$.

Corollary 6.12. *In particular, the condition (6.6.5) and the assertion of Theorem 6.8 hold if*

$$Q(z) = o\left(\log\frac{1}{|z-z_0|}\right) \tag{6.6.6}$$

as $z \to z_0$. More generally, the same holds if

$$q(r) = o\left(\log\frac{1}{r}\right) \tag{6.6.7}$$

as $r \to 0$, where $q(r)$ is the average of the function $Q(z)$ over the circle $|z-z_0| = r$.

Remark 6.5. Choosing in Lemma 6.5 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (6.6.5) by

$$\int\limits_{\varepsilon<|z|<1} \frac{Q(z)\,\mathrm{d}m(z)}{\left(|z|\log\frac{1}{|z|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right), \tag{6.6.8}$$

and (6.6.7) by

$$q(r) = o\left(\log\frac{1}{r}\log\log\frac{1}{r}\right). \tag{6.6.9}$$

Thus, it is sufficient to require instead of (6.6.7) the condition

$$q(r) = O\left(\log\frac{1}{r}\right). \tag{6.6.10}$$

In general, we could give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log\ldots\log 1/t)$. However, we prefer to give here a condition of another type which is often met in mapping theory (see, e.g., [147] and [174]) and which can be obtained directly from Lemma 6.5.

Theorem 6.9. *Let D be a domain in $\mathbb{C}$, $z_0 \in D$, and let $f : D\backslash\{z_0\} \to \mathbb{C}$ be a ring Q-homeomorphism at z_0. If*

$$\int\limits_0^{\varepsilon_0} \frac{\mathrm{d}r}{rq(r)} = \infty, \tag{6.6.11}$$

then f can be extended to D by continuity in $\overline{\mathbb{C}}$.

Proof. Indeed, for the function

$$\psi(t) = \begin{cases} 1/[tq(t)], & t \in (0,\varepsilon_0), \\ 0, & t \in (\varepsilon_0, 1), \end{cases} \tag{6.6.12}$$

we have that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi^2(|z-z_0|)\,\mathrm{d}m(z) \;=\; 2\pi\int\limits_{\varepsilon}^{\varepsilon_0}\frac{\mathrm{d}r}{rq(r)}\,. \tag{6.6.13}$$

Thus, the assertion follows immediately from Lemma 6.5 by the condition (6.6.11). □

Remark 6.6. Thus, by Theorems 2.3 and 2.4, if

$$\int\limits_D \Phi(Q(z))\,\mathrm{d}m(z) \;<\; \infty \tag{6.6.14}$$

for a convex nondecreasing function $\Phi : [0,\infty) \to [0,\infty]$ such that at least one of the conditions (2.8.12)–(2.8.17) holds, then the isolated singular point z_0 of every ring Q-homeomorphism $f : D\backslash\{z_0\} \to \mathbb{C}$ is removable.

In particular, if $f : D\backslash\{z_0\} \to \mathbb{C}$ is a strong ring solution of the Beltrami equation (B), and either

$$Q(z) \;=\; K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{6.6.15}$$

or, more generally,

$$Q(z) \;=\; K^T_\mu(z,z_0) \;=\; \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{6.6.16}$$

satisfies (6.6.14) with the given conditions on Φ, then the isolated singular point z_0 for f is removable under $K_\mu \in L^1_{\rm loc}$.

Thus, Lemma 6.5 is a rich source of various conditions for the removability of isolated singularities of ring Q-homeomorphisms and, in particular, of strong ring solutions of the Beltrami equation (B).

6.7 On Extending Inverse Mappings to the Boundary

We start with the extension to the boundary of the inverse mappings of ring Q-homeomorphisms and strong ring solutions of the Beltrami equation because this problem has a simpler answer than for their direct mappings.

Lemma 6.6. *Let $f : D \to D'$ be a ring Q-homeomorphism at a point $z_0 \in \partial D$ with $Q \in L^1(D)$. If the domain D is locally connected at z_0 and $z_* \in \partial D$, $z_* \neq z_0$, and D' has a weakly flat boundary, then $C(z_0, f) \cap C(z_*, f) = \emptyset$.*

Here, $C(z_0, f)$ denotes the *cluster set* of the mapping f at a point $z_0 \in \partial D$:

$$C(z_0, f) = \{\zeta \in \overline{\mathbb{C}} : \; \zeta = \lim_{k\to\infty} f(z_k), \; z_k \to z_0, \; z_k \in D\}. \tag{6.7.1}$$

Proof. Set $E_0 = C(z_0, f)$, $E_* = C(z_*, f)$ and $\delta = d(z_0, z_*)$. Let us assume that $E_0 \cap E_* \neq \emptyset$.

Since the domain D is locally path connected at the points z_0 and z_*, there exist neighborhoods U_0 and U_* of the points z_0 and z_*, respectively, such that $W_0 = D\cap U_0$ and $W_* = D\cap U_*$ are domains and $U_0 \subset B_0 = B(z_0, \delta/3)$ and $U_* \subset B_* = B(z_*, \delta/3)$. Then, for $r_1 = \delta/3$, $r_2 = 2\delta/3$, any paths γ_0 and γ_* in W_0 and W_*, respectively, $\eta(r) = 3/\delta$, $r \in (r_1, r_2)$, and $\Gamma := \Delta(\gamma_0, \gamma_*, D)$, we obtain

$$M(f\Gamma) \;\le\; \int\limits_{A\cap D} Q(z)\,\eta^2(z)\,{\rm d}m(z) \;\le\; \frac{3^2}{\delta^2}\int\limits_{D} Q(z)\,{\rm d}m(z) \;<\; \infty$$

because $Q \in L^1(D)$. Here, $A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_1| < r_2\}$.

The last estimate contradicts, however, the condition of the weak flatness of $\partial D'$ if there is a point $\zeta_0 \in E_0 \cap E_*$. Indeed, then $\zeta_0 \in \overline{fW_0} \cap \overline{fW_*}$ and there exist paths α_0 and α_* in the domains fW_0 and fW_*, respectively, intersecting any prescribed circles $\partial B(\zeta_0, \rho_1)$ and $\partial B(\zeta_0, \rho_2)$ with small enough radii ρ_1 and ρ_2. Hence, the assumption that $E_0 \cap E_* \neq \emptyset$ was not true. □

By Lemma 6.6, we obtain, in particular, the following conclusion:

Proposition 6.3. *Let a domain $D \subset \mathbb{C}$ be locally connected at ∂D, a domain D' have a weakly flat boundary, and let $f : D \to D'$ be a ring Q-homeomorphism at every point $z \in \partial D$ with $Q \in L^1(D)$. Then the inverse mapping $g = f^{-1} : D' \to D$ admits a continuous extension $\overline{g} : \overline{D'} \to \overline{D}$ in $\overline{\mathbb{C}}$.*

Theorem 6.10. *Let $f : D \to D'$ be a strong ring solution of the Beltrami equation with $K_\mu \in L^1(D)$, where D is locally connected at ∂D and $\partial D'$ is weakly flat. Then the inverse mapping $g = f^{-1} : D' \to D$ admits a continuous extension $\overline{g} : \overline{D'} \to \overline{D}$ in $\overline{\mathbb{C}}$.*

By Lemma 2.20, we also have the following consequence of this theorem:

Corollary 6.13. *If domains D and D' in $\mathbb{C}$ have weakly flat boundaries, then the inverse mapping $g = f^{-1}$ of every strong ring solution $f : D \to D'$ of (B) with $K_\mu \in L^1(D)$ admits a continuous extension $\overline{g} : \overline{D'} \to \overline{D}$ in $\overline{\mathbb{C}}$.*

Remark 6.7. Clearly, it is sufficient in Lemma 6.6, Proposition 6.3, Theorem 6.10, and Corollary 6.13 as well as in all successive theorems to request, instead of the conditions $Q \in L^1(D)$ and $K_\mu \in L^1(D)$, only the integrability of Q and K_μ in a neighborhood of ∂D assuming Q and K_μ to be extended by zero outside of D. Moreover, it is easy to see that these results can be localized in terms of $Q_{z_0}(z)$ and $K_\mu(z, z_0)$, respectively.

Note that, as follows from Proposition 6.2, similar statements are not valid for direct mappings, even for isolated boundary points, if Q and K_μ are integrable with any degree.

6.8 On Extending Direct Mappings to the Boundary

Lemma 6.7. *Let a domain $D \subset \mathbb{C}$ be locally connected at a point $z_0 \in \partial D$, and let $f : D \to D'$ be a ring Q-homeomorphism at z_0 such that $\partial D'$ is strongly accessible at least at one point of the cluster set*

$$C(z_0, f) = \{\zeta \in \overline{\mathbb{C}} : \ \zeta \ = \ \lim_{k\to\infty} f(z_k),\ z_k \to z_0,\ z_k \in D\}, \tag{6.8.1}$$

$Q : D \to [0, \infty]$ is a measurable function satisfying the condition

$$\int\limits_{D(z_0,\varepsilon,\varepsilon_0)} Q(z) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, \mathrm{d}m(z) \ = \ o(I^2_{z_0}(\varepsilon)) \tag{6.8.2}$$

as $\varepsilon \to 0$ where

$$D(z_0, \varepsilon, \varepsilon_0) = \{z \in D : \varepsilon < |z - z_0| < \varepsilon_0\}$$

for some $\varepsilon_0 \in (0,\delta_0)$, $\delta_0 = \delta(z_0) = \sup_{z\in D}|z-z_0|$, *and* $\psi_{z_0,\varepsilon}(t)$ *is a family of nonnegative measurable (by Lebesgue) functions on* $(0,\infty)$ *such that*

$$0 < I_{z_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,\mathrm{d}t < \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,. \tag{6.8.3}$$

Then f *can be extended to the point* z_0 *by continuity in* $\overline{\mathbb{C}}$.

Proof. Let us show that the cluster set $E = C(z_0,f)$ is a singleton. Note that $E \neq \emptyset$ in view of the compactness of $\overline{D'}$ as a subspace of $\overline{\mathbb{C}}$; see, e.g., Remark 3 of Sect. 41 in [142]. By the condition of the lemma, $\partial D'$ is strongly accessible at a point $\zeta_0 \in E$. Assume that there is one more point $\zeta^* \in E$. Let $U = B(\zeta_0, r_0)$ where $0 < r_0 < |\zeta_0 - \zeta^*|$.

In view of the local connectedness of the domain D at the point z_0, there is a sequence of neighborhoods V_m of the point z_0 such that $D_m = D \cap V_m$ are domains and $d(V_m) \to 0$ as $m \to \infty$. Then there exist points ζ_m and $\zeta_m^* \in F_m = fD_m$ which are close enough to ζ_0 and ζ^*, respectively, with $|\zeta_0 - \zeta_m| < r_0$ and $|\zeta_0 - \zeta_m^*| > r_0$. These points ζ_m and ζ_m^* can be joined by paths C_m in the domains F_m. Note that by the construction,

$$C_m \cap \partial B(\zeta_0, r_0) \neq \emptyset$$

in view of the connectedness of C_m.

Thus, by the condition of the strong accessibility, there is a compact set $C \subset D'$ and a number $\delta > 0$ such that

$$M(\Delta(C, C_m; D')) \geq \delta$$

for large m because $\mathrm{dist}(\zeta_0, C_m) \to 0$ as $m \to \infty$. Without loss of generality, we may assume that C is also a continuum. Indeed, as a compact set C, can be covered by a finite number of closed disks in D' each of which can be joined with a point in D' by paths in D'.

The set $K = f^{-1}(C)$ is compact as a continuous image of a compact set and, consequently, $\mathrm{dist}(z_0, K) > 0$. Let us choose positive numbers $\varepsilon_0 < \mathrm{dist}(z_0, K)$ and $\varepsilon \in (0, \varepsilon_0)$ in (6.8.2) and (6.8.3) and set

$$\eta_\varepsilon(t) \;=\; \psi_{z_0,\varepsilon}(t)/I_{z_0}(\varepsilon).$$

Note that by the construction, for large m, $D_m \subset B(z_0, \varepsilon)$ and hence $K_m :\;= f^{-1}(C_m) \subset B(z_0, \varepsilon)$, and thus,

$$M(\Delta(C, C_m; D')) \;\leq \int\limits_{D(z_0,\varepsilon,\varepsilon_0)} Q(z)\cdot \eta_\varepsilon^2(|z-z_0|)\,\mathrm{d}m(z).$$

However, the right-hand side converges to 0 as $\varepsilon \to 0$ by the condition (6.8.2). The obtained contradiction disproves the above assumption that the cluster set E is not a singleton. □

Corollary 6.14. *In particular, if a domain $D \subset \mathbb{C}$ is locally connected at a point $z_0 \in \partial D$, D' is strongly accessible and*

$$\overline{\lim_{\varepsilon \to 0}} \int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z-z_0|)\, \mathrm{d}m(z) \,<\, \infty, \tag{6.8.4}$$

where $\psi(t)$ is a measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon,\varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, \mathrm{d}t \,<\, \infty \qquad \forall\, \varepsilon \in (0,\, \varepsilon_0),$$

and $I(\varepsilon,\varepsilon_0) \to \infty$ as $\varepsilon \to 0$, then every ring Q-homeomorphism $f: D \to D'$ at z_0 can be extended to the point z_0 by continuity in $\overline{\mathbb{C}}$.

Here, we assume that the function Q is extended to be zero outside of D.

Remark 6.8. In other words, the conclusion of Corollary 6.14 holds if the singular integral in (6.8.4) is convergent at the point z_0 in the sense of the principal value at least for one kernel ψ with a nonintegrable singularity at zero. Furthermore, as Lemma 6.7 shows, the conclusion of Corollary 6.14 is still valid if the integral diverges but with the controlled speed

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z-z_0|)\, \mathrm{d}m(z) \,=\, o(I^2(\varepsilon,\varepsilon_0)) \;\text{ as }\; \varepsilon \to 0. \tag{6.8.5}$$

Choosing, in Lemma 6.7, $\psi(t) \equiv 1/t$, we obtain the following theorem:

Theorem 6.11. *Let D and D' be domains in $\mathbb{C}$ such that D is locally connected at $z_0 \in \partial D$ and $\partial D'$ is strongly accessible, and let $f: D \to D'$ be a ring Q-homeomorphism at z_0 with a measurable function $Q: D \to [0,\infty]$ such that*

$$\int\limits_{D(z_0,\varepsilon,\varepsilon_0)} Q(z)\, \frac{\mathrm{d}m(z)}{|z-z_0|^2} \,=\, o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \tag{6.8.6}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 \in (0,\delta_0)$, where $\delta_0 = \delta(z_0) = \sup_{z\in D}|z-z_0|$ and $D(z_0,\varepsilon,\varepsilon_0) = \{z \in D : \varepsilon < |z-z_0| < \varepsilon_0\}$. Then f can be extended to z_0 by continuity in $\overline{\mathbb{C}}$.

Corollary 6.15. *In particular, conclusion of Theorem 6.11 is valid if the singular integral*

$$\int Q(z)\,\frac{\mathrm{d}m(z)}{|z-z_0|^2} \tag{6.8.7}$$

is convergent at the point z_0 in the sense of the principal value.

Here, as in Corollary 6.14, we assume that Q is extended to be zero outside of D.

Corollary 6.16. *In particular, condition (6.8.6) and the assertion of Theorem 6.11 hold if*

$$Q(z) = o\left(\log\frac{1}{|z-z_0|}\right) \tag{6.8.8}$$

as $z \to z_0$. More generally, the same holds if

$$q(r) = o\left(\log\frac{1}{r}\right) \tag{6.8.9}$$

as $r \to 0$, where $q(r)$ is the average of the function $Q(z)$ over the circle $|z-z_0| = r$.

As above, we assume here that Q is extended to be zero outside of D.

Remark 6.9. Arguing as in Remark 6.5, we can show that it is sufficient to require the condition

$$q(r) = O\left(\log\frac{1}{r}\right) \tag{6.8.10}$$

instead of (6.8.9). We are able also to derive a scale of criteria in terms of log using functions $\psi(t)$ of the form $1/(t\log\ldots\log 1/t)$ in Lemma 6.7.

Theorem 6.12. *Let D and D' be domains in $\mathbb{C}$ such that D is locally connected at $z_0 \in \partial D$ and $\partial D'$ is strongly accessible, and let $f: D \to D'$ be a ring Q-homeomorphism at z_0 with a measurable function $Q: D \to [0,\infty]$ such that*

$$\int\limits_0^{\varepsilon_0} \frac{\mathrm{d}r}{rq(r)} = \infty, \tag{6.8.11}$$

where $q(r)$ is the average of $Q(z)$ over the circle $|z-z_0| = r$. Then f can be extended to z_0 by continuity in $\overline{\mathbb{C}}$.

Indeed, as in the proof of Theorem 6.9, it is sufficient to use the function

$$\psi(t) = \begin{cases} 1/[tq(t)], & t \in (0,\varepsilon_0) \\ 0, & t \in (\varepsilon_0,\infty) \end{cases} \tag{6.8.12}$$

in Lemma 6.7.

Further, choosing $\psi_\varepsilon(t) \equiv t\log\frac{1}{t}$, $t \in (0,\delta_0)$ in Lemma 6.7, we obtain by Lemma 5.3 the following theorem:

Theorem 6.13. *Let D and D' be domains in $\mathbb{C}$ such that D is locally connected at a point $z_0 \in \partial D$ and $\partial D'$ is strongly accessible, and let $f : D \to D'$ be a ring Q-homeomorphism at z_0 where $Q : \mathbb{C} \to [0,\infty]$ has finite mean oscillation at the point z_0. Then f can be extended to z_0 by continuity in $\overline{\mathbb{C}}$.*

Finally, combining Theorem 6.13 and Corollary 2.1, we obtain the following statement:

Corollary 6.17. *In particular, if $Q : \mathbb{C} \to [0,\infty]$ is such that*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} Q(z)\, \mathrm{d}m(z) \ < \ \infty, \tag{6.8.13}$$

then any ring Q-homeomorphism $f : D \to D'$ can be extended to the point z_0 by continuity in $\overline{\mathbb{C}}$.

Here, the extension to a point z_0 by continuity in $\overline{\mathbb{C}}$ means that $f(z_0)$ admits ∞.

Remark 6.10. Combining results of Sects. 6.5–6.7, we obtain the corresponding theorems on a homeomorphic extension of ring Q-homeomorphisms to the boundary. The conditions on Q are the same as in Sect. 6.7 plus integrability of Q as in Sect. 6.6 and D is locally connected at all points of ∂D (say if ∂D is weakly flat, see Sect. 6.5) and $\partial D'$ is weakly flat. The similar remarks are true under the integral condition (6.6.14) from Remark 6.6 in Sect. 6.5 assuming that Q, K_μ and K_μ^T is zero outside D.

6.9 Consequences for Quasiextremal Distance Domains

A domain $D \subset \mathbb{C}$ is called a *quasiextremal distance domain*, abbreviated as a *QED domain*, if

$$M(\Delta(E,F;\overline{\mathbb{C}}) \ \le \ K \cdot M(\Delta(E,F;D)) \tag{6.9.1}$$

for some $K \in [1,\infty)$ and for all continua E and F in D; see [92].

Remark 6.11. As is well known (see, e.g., 10.12 in [258]),

$$M(\Delta(E,F;\mathbb{C})) \ \ge \ c\ \log\frac{R}{r} \tag{6.9.2}$$

for all sets E and F in $\mathbb{C}$ intersecting each circle $C(z_0,\rho)$, $\rho \in (r,R)$.

Corollary 6.18. *A* QED *domain D in $\mathbb{C}$ has a weakly flat boundary.*

Corollary 6.19. *A* QED *domain D in $\mathbb{C}$ is locally connected at each boundary point and ∂D is strongly accessible.*

Every QED domain is *quasiconvex*, i.e., each pair of points z_1 and $z_2 \in D$ can be joined by a rectifiable arc γ in D such that

$$s(\gamma) \leq a \cdot |z_1 - z_2|; \tag{6.9.3}$$

see Lemma 2.7 in [92], p. 184. A domain $D \subset \mathbb{C}$ is said to be *uniform* if the inequalities (6.9.3) and

$$\min_{i=1,2} s(\gamma(z_i, z)) \leq b \cdot d(z, \partial D) \tag{6.9.4}$$

hold for some γ and for all $z \in \gamma$ where $\gamma(z_i, z)$ is the part of γ between z_i and z; see [167]. Every uniform domain is a QED domain, but there exist QED domains which are not uniform; see Lemma 2.18 and Remark 2.24 in [92], cf. also [121]. Bounded convex domains provide simple examples of uniform domains.

By Sects. 6.2 and 6.6–6.8 we obtain the following theorems:

Lemma 6.8. *Let f be a ring Q-homeomorphism between QED domains D and D' in $\mathbb{C}$ at a point $z_0 \in \partial D$. If the condition (6.8.2) holds, then there is a limit $f(z)$ as $z \to z_0$ in $\overline{\mathbb{C}}$.*

Corollary 6.20. *If the conditions of Lemma 6.8 hold at every point $z_0 \in \partial D$ and, in addition, $Q \in L^1(D \cap U)$ where U is a neighborhood of ∂D, then f admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

In particular, taking $\psi(t) = 1/t$ and $\psi(t) = 1/t \log \frac{1}{t}$ in (6.8.2), we have, as a consequence of Lemma 6.8, the following theorem (cf. Remark 6.5):

Theorem 6.14. *Let f be a ring Q-homeomorphism between QED domains D and D' in $\mathbb{C}$. If at every point $z \in \partial D$*

$$q(r) = O\left(\log \frac{1}{r}\right) \tag{6.9.5}$$

as $r \to 0$, where $q(r)$ is a mean value of $Q(\zeta)$ over the intersection of the circle $|\zeta - z| = r$ with the domain D, then f extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$.

Corollary 6.21. *In particular, the assertion holds if for every $z \in \partial D$*

$$Q(\zeta) = O\left(\log \frac{1}{|\zeta - z|}\right) \tag{6.9.6}$$

as $\zeta \to z$.

Similarly, choosing $\psi(t) = 1/tq(t)$ in (6.8.2), we have the next consequence of Lemma 6.8; cf. the calculations in the proof of Theorem 6.9:

Corollary 6.22. *The assertion of Theorem 6.14 remains valid if the condition (6.9.5) is replaced by*

$$\int_0^{\varepsilon_0} \frac{dr}{rq(r)} = \infty. \tag{6.9.7}$$

Remark 6.12. Theorem 6.14 and Corollary 6.22 are valid if we take as $q(r)$ the mean of $Q(\zeta)$ over the whole sphere $|\zeta - z| = r$ formally extending $Q(\zeta)$ by zero outside of the domain D.

Choosing $\psi(t) = 1/(t \log 1/t)$ in (6.8.2), we obtain by Lemma 6.8 the following theorem.

Theorem 6.15. *Let f be a ring Q-homeomorphism between QED domains D and D' in $\mathbb{C}$. If $Q(z) \in L^1(D)$ has finite mean oscillation at every point $z_0 \in \partial D$, then f has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Remark 6.13. In particular, this is true for ring Q-homeomorphisms between bounded smooth and convex domains in $\mathbb{C}$.

Corollary 6.23. *If f is a ring Q-homeomorphism of the unit disk $\mathbb{D}$ in $\mathbb{C}$ onto itself where $Q \in L^1(\mathbb{D})$, and*

$$\overline{\lim_{\varepsilon \to 0}} \;\; ⨍_{D(z_0,\varepsilon)} Q(z)\, \mathrm{d}m(z) \; < \; \infty \qquad \forall z_0 \in \partial \mathbb{D} \tag{6.9.8}$$

where $D(z_0, \varepsilon) = D \cap B(z_0, \varepsilon)$, then f admits a homeomorphic extension $\overline{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$.

By Theorem 2.22 in [92], the QED domains coincide with the uniform domains in the class of the so-called finitely connected plane domains. The following example shows that, even among simply connected plane domains, the class of domains with weakly flat boundaries is wider than the class of QED domains.The example is based on Lemma 2.13 in [92] showing, in particular, that QED domains satisfy the condition on doubling measure at every boundary point z_0:

$$\limsup_{r \to 0} \; \frac{|D \cap B(z_0, 2r)|}{|D \cap B(z_0, r)|} \; < \; \infty.$$

The example below just shows that this property on doubling measure is, generally speaking, not valid for domains with weakly flat boundaries.

Example. Consider a simply connected plane domain D of the form

$$D \; = \; \bigcup_{k=1}^{\infty} R_k,$$

where

$$R_k \; = \; \{(x,y) \in \mathbb{R}^2 : \; 0 < x < w_k, \; 0 < y < h_k\}$$

is a sequence of rectangles with quickly decreasing widths $w_k = 2^{-\alpha 2^k} \to 0$ as $k \to \infty$ with $\alpha > 1/\log 2 > 1$ and monotonically increasing heights $h_k = 2^{-1} + \cdots + 2^{-k} \to 1$ as $k \to \infty$.

It is easy to see that D has a weakly flat boundary. This fact is not obvious only for its boundary point $z_0 = (0,1)$. Take as a fundamental system of neighborhoods of the point z_0 the rectangles centered at z_0 of the following form:

$$P_k \ = \ \{(x,y) \in \mathbb{R}^2 : \ |z| < w_k, \ |\zeta - 1| \le 1 - h_{k-1} = 2^{-(k-1)}\},$$

$k = 1,2,\ldots$. Note that

$$P_k \cap D \ = \ \bigcup_{l=k}^{\infty} S_l$$

for all $k > 1$, where

$$S_l \ = \ \{(x,y) \in \mathbb{R}^2 : \ 0 < x < w_l, \ h_{l-1} \le y < h_l\}.$$

Let E and F be an arbitrary pair of continua in D intersecting ∂S_l, i.e., intersecting the horizontal lines $y = h_{l-1}$ and $y = h_l$. Denote by S_l^0 the interior of S_l. Then $\Delta(E,F,S_l^0) \subset \Delta(E,F,D)$ and $\Delta(E,F,S_l^0)$ minorizes the family Γ_l of all paths joining the vertical sides of S_l^0 in S_l^0. Hence,

$$M(\Delta(E,F,D)) \ \ge \ 2^{-l}/w_l \ \ge \ 2^{(\alpha-1)l} \ \to \infty.$$

Thus, the domain D really has a weakly flat boundary.

Now, set $r_k = 1 - h_{k-1} = 2^{-k}(1 + 2^{-1} + \cdots) = 2^{-(k-1)}$ and $B_k = B(z_0, r_k)$. Then

$$\lim_{k\to\infty} \frac{|D \cap P_k|}{|D \cap B_k|} \ = \ 1$$

because $w_k/r_k \le 2^{-(\alpha-1)k} \to 0$. However,

$$|D \cap P_k| \ = \ \sum_{l=k}^{\infty} |S_l| \ = \ \sum_{l=k}^{\infty} w_l \cdot (h_l - h_{l-1}) \ = \ \sum_{l=k}^{\infty} w_l \, 2^{-l}$$

and hence

$$\frac{|D \cap P_k|}{|D \cap P_{k+1}|} \ = \ \frac{\sum\limits_{l=k}^{\infty} w_l 2^{-l}}{\sum\limits_{l=k+1}^{\infty} w_l 2^{-l}} \ = \ \frac{w_k 2^{-k} + \sum\limits_{l=k+1}^{\infty} w_l 2^{-l}}{\sum\limits_{l=k+1}^{\infty} w_l 2^{-l}}$$

$$= \ 1 \ + \ \frac{1}{\sum\limits_{m=1}^{\infty} \frac{w_{k+m}}{w_k} \, 2^{-m}} \ \ge \ 1 \ + \ \frac{1}{\frac{w_{k+1}}{w_k}} = 1 \ + \ \frac{w_k}{w_{k+1}} = 1 \ + \ 2^{\alpha 2^k} \ \to \infty.$$

Consequently,

$$\lim_{k\to\infty} \frac{|D\cap B_k|}{|D\cap B_{k+1}|} = \infty.$$

Thus, the domain D does not have the property on doubling measure at the point $z_0 \in \partial D$, and then D is not a QED domain by Lemma 2.13 in [92]; see also Lemma 3.5 in [165].

6.10 On Singular Null Sets for Extremal Distances

Recall that a closed set $Z \subset \mathbb{C}$ is called a *null set for extremal distances,* abbreviated as a *NED set*, if

$$M(\Delta(E,F;\mathbb{C})) = M(\Delta(E,F;\mathbb{C}\setminus Z)) \tag{6.10.1}$$

for every pair of disjoint continua E and $F \subset \mathbb{C}\setminus Z$.

Here, $\Lambda(Z)$ will denote the *length*, i.e., the one-dimensional Hausdorff measure of Z. As above, we also use here the notion of the *cluster set* of a mapping $f : D \to \overline{\mathbb{C}}$ for a set $Z \subset \overline{D}$,

$$C(Z,f) : = \{\zeta \in \overline{\mathbb{C}} : \zeta = \lim_{k\to\infty} f(z_k),\, z_k \to z_0 \in Z\}. \tag{6.10.2}$$

Remark 6.14. It is known that if $Z \subset \mathbb{C}$ is a NED set, then

$$|Z| = 0 \tag{6.10.3}$$

and Z does not disconnect $\mathbb{C}$ locally, i.e.,

$$\dim Z = 0. \tag{6.10.4}$$

Conversely, if $Z \subset \mathbb{C}$ is closed and

$$\Lambda(Z) = 0, \tag{6.10.5}$$

then X is a NED set; see [262].

Note that the complements of NED sets in $\mathbb{C}$ are a very particular case of QED domains considered in the previous section. Thus, arguing locally, we obtain, as above, the following statement:

Lemma 6.9. *Let D be a domain in $\mathbb{C}$ and $Z \subset D$, and let f be a ring Q-homeomorphism of $D\setminus Z$ into $\mathbb{C}$. Suppose that Z and $C(Z,f)$ are NED sets and Q is integrable in a neighborhood of the set Z. If the condition (6.8.2) holds at every point $z_0 \in Z$, then f has a homeomorphic extension to D in $\overline{\mathbb{C}}$.*

Choosing $\psi(t) = 1/(t\log 1/t)$ in (6.8.2), we have, as a consequence of Lemma 6.9, the following theorem:

Theorem 6.16. *If $Q \in L^1_{\mathrm{loc}}(D)$ has finite mean oscillation at each point of a NED set $Z \subset D$, then every ring Q-homeomorphism f of $D\backslash Z$ into $\overline{\mathbb{C}}$ with a NED set $C(Z, f)$ has a homeomorphic extension to D in $\overline{\mathbb{C}}$.*

In view of Remark 6.14, we obtain the following consequence of Theorem 6.16:

Corollary 6.24. *In particular, the assertion of Theorem 6.16 holds if Z is a closed subset of D with*

$$\Lambda(Z) = \Lambda(C(Z, f)) = 0 \tag{6.10.6}$$

and $Q \in L^1_{\mathrm{loc}}(D)$ has finite mean oscillation at every point $z \in Z$.

In particular, by Corollary 2.1, we come to the next consequence.

Corollary 6.25. *If all points of a closed set $Z \subset D$ with the condition (6.10.6) are points of Lebesgue for the function $Q \in L^1_{\mathrm{loc}}(D)$, then the ring Q-homeomorphism f of $D\backslash Z$ into $\overline{\mathbb{C}}$ admits a homeomorphic extension to D in $\overline{\mathbb{C}}$.*

By Lemma 6.9 under $\psi(t) = 1/t$, we have also the next statement.

Corollary 6.26. *If the singular integral*

$$\int\limits_U \frac{Q(\zeta) - Q(z)}{|\zeta - z|^2}\,\mathrm{d}m(z) \tag{6.10.7}$$

is convergent for each ζ in a closed set $Z \subset D$ over a neighborhood U of the set Z, then under the condition (6.10.6), every ring Q-homeomorphism f of $D\backslash Z$ into $\overline{\mathbb{C}}$ has homeomorphic extension to D in $\overline{\mathbb{C}}$.

In the same way, analogies of all other theorems in Sects. 6.5 and 6.7–6.9 can be obtained by Lemma 6.9, too.

Remark 6.15. It is easy to show that all results of the previous sections in this chapter are applicable to the strong ring solutions of the Beltrami equation (B) in the next chapter, either with

$$Q(z) \;=\; K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{6.10.8}$$

or, more generally, with

$$Q(z) \;=\; K^T_\mu(z, z_0) \;=\; \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}\,. \tag{6.10.9}$$

Chapter 7
Strong Ring Solutions of Beltrami Equations

In this chapter we give a series of criteria for the existence of strong ring solutions, in particular, in terms of finite mean oscillation majorants for tangential dilatations. Moreover, we derive an extension of the well-known Lehto existence theorem and show that the latter implies the main known and many advanced results on the existence of ACL homeomorphic solutions for the Beltrami equations with degeneration.

7.1 Introduction

Recall that an ACL homeomorphism $f : D \to \mathbb{C}$ is called a *strong ring solution* of the Beltrami equation (B) with a complex coefficient μ if f satisfies (B) a.e., $f^{-1} \in W^{1,2}_{\mathrm{loc}}$, and f is a ring Q-homeomorphism at every point $z_0 \in \overline{D}$ with $Q_{z_0}(z) = K^T_\mu(z, z_0)$; see (6.1.6) and (6.1.14). Thus, the whole theory of ring Q-homeomorphisms developed in the previous chapter can be applied to the study of the boundary behavior as well as the removability of singularities of the strong ring solutions.

We show that the strong ring solutions exist for wide classes of degenerate Beltrami equations. The difference between the notion of a ring solution and the notion of a strong ring solution is that, while the centers z_0 are assumed to belong to the domain D in the notion of a ring solution, in the case of a strong ring solution, z_0 is assumed to be in the closure of D relative to $\mathbb{C}$.

The notion of a strong ring solution for the Beltrami equation is closely related to the concept of moduli with weights essentially due to Andreian Cazacu; see, e.g., [15] and [16], cf. also [179, 180] and [250]. The work [215] was devoted to existence of ACL homeomorphic solutions satisfying additional moduli conditions at inner points of a domain. In the papers [220] and [221], a base of the theory for the strong ring solutions with moduli conditions at boundary points was given. As in [99], we formulate existence criteria not only in terms of the maximal dilatation but also in

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terms of the so-called tangential dilatation also due to Andreian Cazacu; see, e.g., [14, 16, 18], cf. the corresponding terms and notations in [13, 66, 99, 147, 198, 214] and [215].

The classical case of the uniformly elliptic Beltrami equations was investigated long ago; see, e.g., [9, 30, 44] and [152]. The existence problem for degenerate Beltrami equations is currently an active area of research. It has been studied extensively, and many contributions have been made; and see, e.g., [48, 54–57, 66, 70, 99, 117, 133, 147, 159, 160, 174, 185, 210–223, 254], and [271], and see also the survey [238]. The goal here is to obtain a general principle on the existence of strong ring solutions for the degenerate Beltrami equations (B) (see Lemma 7.1) and to show that our extension of the Lehto existence theorem (see Theorem 7.5) has as corollaries the main known existence theorems as well as a series of more advanced theorems for the Beltrami equations with integral constraints; see further historical comments in Sect. 7.7.

Note that the condition $K_\mu \in L^1_{\text{loc}}(D)$ below is necessary for a homeomorphic ACL solution f of (B) to have the property $g = f^{-1} \in W^{1,2}_{\text{loc}}(f(D))$ because this property implies that

$$\int\limits_C K_\mu(z)\,\mathrm{d}m(z) \le 4\int\limits_C \frac{\mathrm{d}m(z)}{1-|\mu(z)|^2} = 4\int\limits_{f(C)} |\partial g|^2\,\mathrm{d}u\mathrm{d}v < \infty$$

for every compact set $C \subset D$. The change of variables is correct here, say by Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [152]; cf. also I.C(3) in [9].

In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty(D)$, every ACL homeomorphic solution f of the Beltrami equation (B) is in the class $W^{1,2}_{\text{loc}}(D)$ together with its inverse mapping f^{-1}, and hence, f is a strong ring solution of (B) by Theorem 6.1. In the case $\|\mu\|_\infty = 1$ with $K_\mu \le Q \in \text{BMO}$, again $f^{-1} \in W^{1,2}_{\text{loc}}(f(D))$ and f belongs to $W^{1,s}_{\text{loc}}(D)$ for all $1 \le s < 2$, but not necessarily to $W^{1,2}_{\text{loc}}(D)$; see Chap. 5.

7.2 General Existence Lemma and Corollaries

The following lemma and corollary serve as the main tool in obtaining many criteria of existence of strong ring solutions for the Beltrami equations:

Lemma 7.1. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 < \operatorname{dist}(z_0, \partial D)$ and a one-parameter family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, such that*

$$0 \,<\, I_{z_0}(\varepsilon) \,:=\, \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,\mathrm{d}t \,<\, \infty, \tag{7.2.1}$$

and such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,\mathrm{d}m(z) \;=\; o(I^2_{z_0}(\varepsilon)) \tag{7.2.2}$$

as $\varepsilon\to 0$. *Then the Beltrami equation (B) has a strong ring solution.*

Proof. Fix z_1 and z_2 in D. For $n\in\mathbb{N}$, define $\mu_n : D\to\mathbb{C}$ by letting $\mu_n(z)=\mu(z)$ if $|\mu(z)|\le 1-1/n$ and 0 otherwise. Let $f_n : D\to\mathbb{C}$ be a homeomorphic ACL solution of the Beltrami equation (B), with μ_n instead of μ, which fixes z_1 and z_2. Such f_n exists by the well-known existence theorem in the nondegenerate case; see, e.g., [9, 30, 44] and [152]. By Lemma 2.13 and Theorem 6.1, in view of (7.2.2), the sequence f_n is equicontinuous, and hence, by the Arzela–Ascoli theorem (see, e.g., [73], p. 382, and [74], p. 267), it has a subsequence, denoted again by f_n, which converges locally uniformly to some nonconstant mapping f in D. Then by Corollary 2.3 from Sect. 2.6, f is a homeomorphic ACL solution of (B). Moreover, by Theorems 6.1 and 6.2, f is a ring Q-homeomorphism with $Q(z)=K^T_\mu(z,z_0)$ at every point $z_0\in\overline{D}$. The inclusion $f^{-1}\in W^{1,2}_{\mathrm{loc}}$ follows by Corollary 2.4 from Sect. 2.4. □

Remark 7.1. By Propositions 2.4 and 2.5 from Sect. 2.4, $f_\mu\in W^{1,1}_{\mathrm{loc}}$ and has the N^{-1} property and f_μ is regular, i.e., differentiable with $J_{f^\mu}(z)>0$ a.e.

Corollary 7.1. *Let* $\mu : D\to\mathbb{C}$ *be a measurable function with* $|\mu(z)|<1$ *a.e.,* $K_\mu\in L^1_{\mathrm{loc}}(D)$, *and let* $\psi:(0,\infty)\to(0,\infty)$ *be a measurable function such that for all* $0<t_1<t_2<\infty$

$$0\;<\;\int\limits_{t_1}^{t_2}\psi(t)\,\mathrm{d}t<\;\infty,\qquad \int\limits_{0}^{t_2}\psi(t)\,\mathrm{d}t\;=\;\infty. \tag{7.2.3}$$

Suppose that for every $z_0\in D$ *there is* $\varepsilon_0<\mathrm{dist}\,(z_0,\partial D)$ *such that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z)\cdot\psi^2(|z-z_0|)\,\mathrm{d}m(z)\;\le\;O\left(\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,\mathrm{d}t\right) \tag{7.2.4}$$

as $\varepsilon\to 0$. *Then the Beltrami equation (B) has a strong ring solution.*

Lemma 7.1 yields the following theorem by choosing:

$$\psi_{z_0,\varepsilon}(t)\;=\;\frac{1}{t\log\frac{1}{t}}; \tag{7.2.5}$$

see also Lemma 5.3.

Theorem 7.1. *Let* $\mu : D\to\mathbb{C}$ *be a measurable function with* $|\mu(z)|<1$ *a.e. and* $K_\mu\in L^1_{\mathrm{loc}}(D)$. *Suppose that every point* $z_0\in D$ *has a neighborhood* U_{z_0} *such that*

$$K^T_\mu(z,z_0)\;\le\;Q_{z_0}(z)\quad \textit{a.e.} \tag{7.2.6}$$

for some function $Q_{z_0}(z)$ of finite mean oscillation at the point z_0 in the variable z. Then the Beltrami equation (B) has a strong ring solution.

The following theorem is a consequence of Theorem 7.1 and Corollary 2.1:

Theorem 7.2. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}(D)$. Suppose that at every $z_0 \in D$*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\, \mathrm{d}m(z) \ < \ \infty. \tag{7.2.7}$$

Then the Beltrami equation (B) has a strong ring solution f_μ.

The following theorem is an important particular case of Theorem 7.1.

Theorem 7.3. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \ \leq \ Q(z) \ \in \ \mathrm{FMO}. \tag{7.2.8}$$

Then the Beltrami equation (B) has a strong ring solution.

Since every strong ring solution is a homeomorphic ACL solution and since every BMO function is in FMO, the theorem strengthens earlier results about the existence of ACL homeomorphic solutions of the Beltrami equation when the conditions involve majorants of the classes BMO and FMO in the previous sections.

Corollary 7.2. *If*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} \frac{1+|\mu(z)|}{1-|\mu(z)|}\, \mathrm{d}m(z) \ < \ \infty \tag{7.2.9}$$

at every $z_0 \in D$, then the Beltrami equation (B) has a strong ring solution.

Applying Lemma 7.1 with $\psi(t) = 1/t$, we have also the following statement which is formulated in terms of the logarithmic mean (see (6.5.36)) of $K^T_\mu(z,z_0)$ over the annuli $A(\varepsilon) = \{z \in \mathbb{C} : \varepsilon < |z-z_0| < \varepsilon_0\}$ for a fixed $\varepsilon_0 = \delta(z_0) \leq \mathrm{dist}\,(z_0, \partial D)$:

Theorem 7.4. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. If at every point $z_0 \in D$ the logarithmic mean of K^T_μ over $A(\varepsilon)$ does not converge to ∞ as $\varepsilon \to 0$, i.e.,*

$$\liminf_{\varepsilon\to 0} \ M^{K^T_\mu}_{\log}(\varepsilon) \ < \ \infty, \tag{7.2.10}$$

then the Beltrami equation (B) has a strong ring solution.

Corollary 7.3. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Denote by $q^T_{z_0}(t)$ the mean of $K^T_\mu(z,z_0)$ over the circle $C = \{z \in \mathbb{C} : |z-z_0| = t\}$. If*

$$\int_0^{\delta(z_0)} q_{z_0}^T(t)\,\frac{dt}{t} \;<\; \infty \tag{7.2.11}$$

at every point $z_0 \in D$ *for some* $\delta(z_0) > 0$, *then (B) has a strong ring solution.*

7.3 Extension of Lehto's Existence Theorem

Lehto considers in [147] degenerate Beltrami equations in the special case where the *singular set* S_μ,

$$S_\mu = \left\{ z \in \mathbb{C} : \lim_{\varepsilon \to 0} \|K_\mu\|_{L^\infty(B(z,\varepsilon))} \;=\; \infty \right\}, \tag{7.3.1}$$

of the complex coefficient μ in the Beltrami equation (B) is of measure zero and shows that if for every $z_0 \in \mathbb{C}$ and every r_1 and $r_2 \in (0,\infty)$ the integral

$$\int_{r_1}^{r_2} \frac{dr}{r(1+q_{z_0}^T(r))}\,, \quad r_2 > r_1 \tag{7.3.2}$$

is positive and tends to ∞ as either $r_1 \to 0$ or $r_2 \to \infty$ where

$$q_{z_0}^T(r) \;=\; \frac{1}{2\pi} \int_0^{2\pi} \frac{|1-e^{-2i\vartheta}\mu(z_0+re^{i\vartheta})|^2}{1-|\mu(z_0+re^{i\vartheta})|^2}\; d\vartheta, \tag{7.3.3}$$

then there exists a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is ACL in $\mathbb{C} \setminus S_\mu$ and satisfies (B) a.e. Note that the integrand in (7.3.3) is the tangential dilatation $K_\mu^T(z,z_0)$.

We present now a generalization and strengthening of Lehto's existence theorem which enables us to derive a series of other existence theorems, some of which are known and some are new. In this extension, we prove the existence of a strong ring solution in a domain $D \subset \mathbb{C}$ which, by the definition, is ACL in D and not only in $D \setminus S_\mu$. Note that the situation where $S_\mu = D$ is possible here and that the condition (7.3.4) is a little weaker than the Lehto condition (7.3.2) because $q_{z_0}^T(r)$ can be arbitrarily close to 0. See Remark 7.2 below for the case where $\infty \in D$, too.

Theorem 7.5. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$. Suppose that at every point $z_0 \in D$*

$$\int_0^{\delta(z_0)} \frac{dr}{r q_{z_0}^T(r)} \;=\; \infty, \tag{7.3.4}$$

where $\delta(z_0) < \operatorname{dist}(z_0, \partial D)$ and $q_{z_0}^T(r)$ is the mean of $K_\mu^T(z,z_0)$ over $|z-z_0| = r$. Then the Beltrami equation (B) has a strong ring solution f_μ.

A version of this theorem for the ring solutions was first published in the preprint [214], and then in [215], see also [165]. The advanced version for the strong ring solutions was published in [220].

Note that already in the work [174], the existence of homeomorphic solutions to the Beltrami equation (B) in the class $f_\mu \in W^{1,s}_{\mathrm{loc}}$, $s = 2p/(1+p)$ was established, under the condition (7.3.4) with $K_\mu \in L^p_{\mathrm{loc}}$, $p > 1$, instead of $K^T_\mu(z,z_0)$ (the case $p = 1$ is covered thanks to a new convergence theorem in Sect. 2.6, Theorem 2.1). The Miklyukov–Suvorov result was newly discovered in the paper [66] whose author thanks Professor F.W. Gehring but does not mention [174]; see the English translation of this paper in our appendixes. Perhaps the work [174] remains unknown even to the leading experts in the West because we have found no reference to this work in the latest monograph on the Beltrami equations under the discussion of the Lehto condition; see Theorem 20.9.4 in [26].

Theorem 7.5, together with lemmas in Sects. 2.8 and 2.9, is the main tool for deriving all theorems on existence of strong ring solutions for Beltrami equations with various integral conditions in the next section.

Proof. Theorem 7.5 follows from Lemma 7.1 by specially choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \;\equiv\; \psi_{z_0}(t) \;:=\; \begin{cases} 1/[tq^T_{z_0}(t)], & t \in (0,\varepsilon_0), \\ 0, & \text{otherwise}, \end{cases} \tag{7.3.5}$$

where $\varepsilon_0 = \delta(z_0)$. □

Corollary 7.4. *If $K_\mu \in L^1_{\mathrm{loc}}(D)$, and at every point $z_0 \in D$*

$$q^T_{z_0}(r) \;=\; O\left(\log\frac{1}{r}\right) \quad as \quad r \to 0, \tag{7.3.6}$$

then the Beltrami equation (B) has a strong ring solution.

Since $K^T_\mu(z,z_0) \le K_\mu(z)$, we obtain, as a consequence of Theorem 7.5, the following result which is due to Miklyukov and Suvorov [174] for the case $K_\mu \in L^p_{\mathrm{loc}}(D)$, $p > 1$:

Corollary 7.5. *If $K_\mu \in L^p_{\mathrm{loc}}(D)$ for $p \ge 1$ and (7.3.4) holds for $K_\mu(z)$ instead of $K^T_\mu(z,z_0)$ for every point $z_0 \in D$, then the Beltrami equation (B) has a $W^{1,s}_{\mathrm{loc}}(D)$ homeomorphic solution, where $s = 2p/(p+1)$.*

Remark 7.2. By Proposition 2.4, a strong ring solution of the Beltrami equation (B) belongs just to the class $W^{1,s}_{\mathrm{loc}}(D)$, where $s = 2p/(p+1)$ if $K_\mu \in L^p_{\mathrm{loc}}(D)$ for $p \ge 1$.

All the above theorems can be extended to the case where $\infty \in D \subset \overline{\mathbb{C}}$ in the standard way, if one replaces (7.3.4) by the following condition at ∞:

$$\int\limits_{\delta}^{\infty} \frac{\mathrm{d}r}{rq(r)} \;=\; \infty \tag{7.3.7}$$

where $\delta > 0$ and

$$q(r) \;=\; \frac{1}{2\pi} \int_0^{2\pi} \frac{|1 - \mathrm{e}^{-2\mathrm{i}\vartheta}\mu(r\mathrm{e}^{\mathrm{i}\vartheta})|^2}{1 - |\mu(r\mathrm{e}^{\mathrm{i}\vartheta})|^2} \;\mathrm{d}\vartheta. \tag{7.3.8}$$

In this case, there exists a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution $f = f_\mu$ in D with $f(\infty) = \infty$ and $f_\mu^{-1} \in W^{1,2}_{\mathrm{loc}}(f(D))$. Here, $f \in W^{1,1}_{\mathrm{loc}}(D)$ in D means that $f \in W^{1,1}_{\mathrm{loc}}(D \setminus \{\infty\})$ and that $f^*(z) = 1/\overline{f(1/\overline{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(D))$ has a similar meaning. The FMO conditions can be also extended in the natural way to ∞; see, e.g., [217].

7.4 Sufficient Integral Conditions for Solvability

The following existence theorem is obtained immediately from Theorems 2.4 and 7.5:

Theorem 7.6. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where*

$$\int_{U_{z_0}} \Phi_{z_0}(K^T_\mu(z, z_0)) \,\mathrm{d}m(z) \;<\; \infty \tag{7.4.1}$$

for a nondecreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty]$ such that

$$\int_{\Delta(z_0)}^{\infty} \frac{\mathrm{d}\tau}{\tau \Phi^{-1}_{z_0}(\tau)} \;=\; \infty \tag{7.4.2}$$

for some $\Delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (B) has a strong ring solution.

Proof. Let the closure of a disk $B(z_0, \rho)$ belong to the neighborhood U_{z_0}. Then we obtain by Theorem 2.4 applied to $Q(\zeta) = K^T_\mu(z_0 + \rho\zeta, z_0)$, $\zeta \in \mathbb{D}$, and $\Phi(t) = \Phi_{z_0}(t)$ that

$$\int_0^{\rho} \frac{\mathrm{d}r}{r q^T_{z_0}(r)} \;=\; \infty, \tag{7.4.3}$$

where $q^T_{z_0}(r)$ is the mean value of $K^T_\mu(z, z_0)$ over the circle $|z - z_0| = r$. Thus, we have the desired conclusion by Theorem 7.5. □

Remark 7.3. Note that the additional condition $\Delta(z_0) > \Phi_{z_0}(0)$ is essential; see also Remark 2.12. In fact, only the degree of convergence $\Phi^{-1}_{z_0}(\tau) \to \infty$ as $\tau \to \infty$ is important since it has the same degree of convergence $\Phi_{z_0}(t) \to \infty$ as $t \to \infty$.

Corollary 7.6. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. Suppose that*

$$\int_D \Phi(K_\mu(z))\, \mathrm{d}m(z) \;<\; \infty \tag{7.4.4}$$

for a nondecreasing convex function $\Phi : [0,\infty] \to [0,\infty]$ such that

$$\int_\Delta^\infty \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)} \;=\; \infty \tag{7.4.5}$$

for some $\Delta > \Phi(0)$. Then the Beltrami equation (B) has a strong ring solution.

Remark 7.4. Applying transformations $\alpha\cdot\Phi+\beta$ with $\alpha > 0$ and $\beta \in \mathrm{Re}$, we may assume without loss of generality that $\Phi(t) = \Phi(1) = 1$ for all $t \in [0,1]$, and thus $\Phi(0) = \Phi(1) = 1$ in Theorem 7.6 and its corollaries below.

Many other criteria for the existence of strong ring solutions for the Beltrami equation (B) formulated below follow from Theorems 2.3 and 7.6.

Corollary 7.7. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. If the condition (7.4.1) holds at every point $z_0 \in D$ with a nondecreasing convex function $\Phi_{z_0} : [0,\infty) \to [0,\infty)$ such that*

$$\int_{\Delta(z_0)}^\infty \log\Phi_{z_0}(t)\,\frac{\mathrm{d}t}{t^2} \;=\; \infty \tag{7.4.6}$$

for some $\Delta(z_0) > 0$, then the Beltrami equation (B) has a strong ring solution.

Corollary 7.8. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. If the condition (7.4.1) holds at every point $z_0 \in D$ for a continuous nondecreasing convex function $\Phi_{z_0} : [0,\infty) \to [0,\infty)$ such that*

$$\int_{\Delta(z_0)}^\infty (\log\Phi_{z_0}(t))'\,\frac{\mathrm{d}t}{t} \;=\; \infty \tag{7.4.7}$$

for some $\Delta(z_0) > 0$, then (B) has a strong ring solution.

Corollary 7.9. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\mathrm{loc}}$. If the condition (7.4.1) holds at every point $z_0 \in D$ for $\Phi_{z_0} = \exp H_{z_0}$ where H_{z_0} is nonconstant, nondecreasing, and convex, then (B) has a strong ring solution.*

Corollary 7.10. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. If the condition (7.4.1) holds at every point $z_0 \in D$ for $\Phi_{z_0} = \exp H_{z_0}$ with a twice continuously differentiable increasing function H_{z_0} such that*

$$H''_{z_0}(t)) \geq 0 \qquad \forall\, t \geq t(z_0) \in [0,\infty), \tag{7.4.8}$$

then (B) has a strong ring solution.

Theorem 7.7. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$ such that*

$$\int\limits_D \Phi(K_\mu(z))\, \mathrm{d}m(z) < \infty, \tag{7.4.9}$$

where $\Phi : [0,\infty) \to [0,\infty]$ is nondecreasing and convex such that

$$\int\limits_\Delta^\infty \log \Phi(t)\, \frac{\mathrm{d}t}{t^2} = \infty \tag{7.4.10}$$

for some $\Delta > 0$. Then the Beltrami equation (B) has a strong ring solution.

Corollary 7.11. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{\text{loc}}$. If the condition (7.4.9) holds with a nondecreasing convex function Φ : $[0,\infty) \to [0,\infty)$ such that*

$$\int\limits_{t_0}^\infty (\log \Phi(t))'\, \frac{\mathrm{d}t}{t} = \infty \tag{7.4.11}$$

for some $t_0 > 0$, then (B) has a strong ring solution.

Corollary 7.12. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (7.4.9) holds for $\Phi = \mathrm{e}^H$ where H is nonconstant, nondecreasing, and convex, then (B) has a strong ring solution.*

Corollary 7.13. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (7.4.9) holds for $\Phi = \mathrm{e}^H$ where H is twice continuously differentiable, increasing and*

$$H''(t) \geq 0 \qquad \forall\, t \geq t_0 \in [1,\infty), \tag{7.4.12}$$

then (B) has a strong ring solution.

Note that among twice continuously differentiable functions, the condition (7.4.12) is equivalent to the convexity of $H(t)$, $t \geq t_0$; cf. Corollary 7.12. Of course, the convexity of $H(t)$ implies the convexity of $\Phi(t) = \mathrm{e}^{H(t)}$, $t \geq t_0$, because the function $\exp x$ is convex. However, in general, the convexity of Φ does not imply the convexity of $H(t) = \log \Phi(t)$, and it is known that the convexity of $\Phi(t)$ in

(7.4.9) is not sufficient for the existence of ACL homeomorphic solutions of the Beltrami equation. There exist examples of the complex coefficients μ such that $K_\mu \in L^p$ with an arbitrarily large $p \geq 1$ for which the Beltrami equation (B) has no ACL homeomorphic solutions; see, e.g., [212].

Remark 7.5. Theorem 7.5 is extended by us to the case where $\infty \in D \subset \overline{\mathbb{C}}$ in the standard way by replacing (7.3.4) by the following condition at ∞ :

$$\int_{\delta}^{\infty} \frac{\mathrm{d}r}{rq(r)} = \infty \tag{7.4.13}$$

where $\delta > 0$ and

$$q(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|1 - \mathrm{e}^{-2\mathrm{i}\vartheta}\mu(r\mathrm{e}^{\mathrm{i}\vartheta})|^2}{1 - |\mu(r\mathrm{e}^{\mathrm{i}\vartheta})|^2}\, \mathrm{d}\vartheta. \tag{7.4.14}$$

In this case, there exists a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution f of (B) in D with $f(\infty) = \infty$ and $f^{-1} \in W^{1,2}_{\mathrm{loc}}$. Here, $f \in W^{1,1}_{\mathrm{loc}}$ in D means that $f \in W^{1,1}_{\mathrm{loc}}$ in $D \setminus \{\infty\}$ and that $f^*(z) = 1/\overline{f(1/\overline{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W^{1,2}_{\mathrm{loc}}$ has a similar meaning.

Similarly, the integral condition (7.4.1) is replaced at ∞ by the following condition:

$$\int_{|z|>\delta} \Phi_\infty(K^T_\mu(z,\infty))\, \frac{\mathrm{d}m(z)}{|z|^4} < \infty, \tag{7.4.15}$$

where $\delta > 0$, Φ_∞ satisfies the conditions of either Theorem 7.6 or equivalent conditions from Theorem 2.3 and

$$K^T_\mu(z,\infty) = \frac{\left|1 - \frac{\overline{z}}{z}\mu(z)\right|^2}{1 - |\mu(z)|^2}\,. \tag{7.4.16}$$

We may assume in all the above theorems that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex on the whole segments $[0,\infty]$ and $[1,\infty]$, respectively, but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$. Indeed, every nondecreasing function Φ : $[1,\infty] \to [0,\infty]$ which is convex on a segment $[T,\infty]$, $T \in (0,\infty)$, can be replaced by a nondecreasing convex function $\Phi_T : [0,\infty] \to [0,\infty]$ in the following way: We set $\Phi_T(t) \equiv 0$ for all $t \in [0,T]$, $\Phi(t) = \varphi(t)$, $t \in [T,T_*]$, and $\Phi_T \equiv \Phi(t)$, $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,T)$ and touching the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \geq T$. For such a function, we have by the construction that $\Phi_T(t) \leq \Phi(t)$ for all $t \in [1,\infty]$ and $\Phi_T(t) = \Phi(t)$ for all $t \geq T_*$.

7.5 Necessary Integral Conditions for Solvability

The main idea for the proof of the following statement under smooth increasing functions Φ with the additional condition that $t(\log\Phi)' \geq 1$ is due to Iwaniec and Martin; see Theorem 3.1 in [117], cf. also Theorem 11.2.1 in [116] and Theorem 20.3.1 in [26]. We obtain the same conclusion in Lemma 7.2 and Theorem 7.8 below without these smooth conditions. Moreover, by Theorem 2.3, the same conclusion concerns all conditions (2.8.13)–(2.8.17).

Theorem 7.8. *Let $\Phi : [0,\infty] \to [0,\infty]$ be such a nondecreasing convex function that, for every measurable function $\mu : \mathbb{D} \to \mathbb{D}$ satisfying the condition*

$$\int\limits_{\mathbb{D}} \Phi(K_\mu(z))\, \mathrm{d}m(z) \ < \ \infty, \tag{7.5.1}$$

the Beltrami equation (B) has a homeomorphic ACL solution. Then there is $\delta > 0$ such that

$$\int\limits_{\delta}^{\infty} \log\Phi(t)\, \frac{\mathrm{d}t}{t^2} \ = \ \infty. \tag{7.5.2}$$

It is evident that the function $\Phi(t)$ in Theorem 7.8 is not constant on $[0,\infty)$ because in the contrary case we would have no real restrictions for K_μ from (7.5.1) except $\Phi(t) \equiv \infty$ when the class of such μ is empty. Moreover, by the well-known criterion of convexity (see, e.g., Proposition 5 in I.4.3 of [53]) the inclination $\Phi(t)/t$ is nondecreasing. Hence, the proof of Theorem 7.8 is reduced to the following statement:

Lemma 7.2. *Let a function $\Phi : [0,\infty] \to [0,\infty]$ be nondecreasing and*

$$\Phi(t) \ \geq \ C\cdot t \qquad \forall\, t \ \geq \ T \tag{7.5.3}$$

for some $C > 0$ and $T \in (1,\infty)$. If the Beltrami equations (B) have ACL homeomorphic solutions for all measurable functions $\mu : \mathbb{D} \to \mathbb{D}$ satisfying the condition (7.5.1), then (7.5.2) holds for some $\delta > 0$.

Remark 7.6. Note that the Iwaniec–Martin condition $t(\log\Phi)' \geq 1$ implies the condition (7.5.3) with $C = \Phi(T)/T$. Note also that if we take in the construction of Lemma 7.4 below $\beta_{n+1} = \varepsilon_n \to 0$ as $n \to \infty$, $\alpha_{n+1} = b_n e^{b_n\gamma_n} - \varepsilon_n\gamma_n$ and $\gamma^*_{n+1} = b_n e^{b_n\gamma_n}/\varepsilon_n$, then $\Phi(\gamma^*_{n+1})/\gamma^*_{n+1} \leq 2\varepsilon_n$, and we obtain examples of absolutely continuous increasing functions Φ with $\Phi(t) \to \infty$ as $t \to \infty$ satisfying the condition (7.5.2) and simultaneously

$$\liminf_{t\to\infty} \frac{\Phi(t)}{t} \ = \ 0. \tag{7.5.4}$$

Thus, conditions of the type (7.5.3) are independent of the conditions (2.8.12)–(2.8.17).

Proof of Lemma 7.2. Let us assume that the condition (7.5.2) does not hold for any $\delta > 0$. Set $t_0 = \sup_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(t) > 0$ for all $t \in [0,\infty]$. Then for all $\delta > t_0$,

$$\int\limits_{\delta}^{\infty} \log \Phi(t)\, \frac{\mathrm{d}t}{t^2} \;<\; \infty. \tag{7.5.5}$$

With no loss of generality, applying the linear transformation $\alpha\Phi + \beta$ with $\alpha = 1/C$ and $\beta = T$, we may assume by (7.5.3) that

$$\Phi(t) \;\geq\; t \qquad\qquad \forall\, t \in [0,\infty). \tag{7.5.6}$$

Of course, we may also assume that $\Phi(t) = t$ for all $t \in [0,1)$ because the values of Φ in $[0,1)$ give no information on K_μ in (7.5.1). Finally, by (7.5.5), we have that $\Phi(t) < \infty$ for every $t \in [0,\infty)$.

Now, note that the function $\Psi(t) :\, = t\Phi(t)$ is strictly increasing, $\Psi(1) = \Phi(1)$ and $\Psi(t) \to \infty$ as $t \to \infty$, $\Psi(t) < \infty$ for every $t \in [0,\infty)$. Hence, the functional equation

$$\Psi(K(r)) \;=\; \left(\frac{\gamma}{r}\right)^2 \qquad\qquad \forall\, r \;\in\; (0,1], \tag{7.5.7}$$

where $\gamma = \Phi^{1/2}(1) \geq 1$, is well solvable with $K(1) = 1$ and a continuous nonincreasing function $K : (0,1] \to [1,\infty)$ such that $K(r) \to \infty$ as $r \to 0$. Taking the logarithm in (7.5.7), we have that

$$2\log\, r \;+\; \log\, K(r) \;+\; \log\, \Phi(K(r)) \;=\; 2\, \log\, \gamma,$$

and by (7.5.6) we obtain that

$$\log\, r \;+\; \log\, K(r) \;\leq\; \log\, \gamma\,,$$

i.e.,

$$K(r) \;\leq\; \frac{\gamma}{r}\,. \tag{7.5.8}$$

Then by (7.5.7),

$$\Phi(K(r)) \;\geq\; \frac{\gamma}{r},$$

and hence, by (2.8.2),

$$K(r) \;\geq\; \Phi^{-1}\left(\frac{\gamma}{r}\right).$$

Thus,

$$I(t) \;:\; = \int\limits_0^t \frac{\mathrm{d}r}{rK(r)} \;\leq\; \int\limits_0^t \frac{\mathrm{d}r}{r\Phi^{-1}\left(\frac{\gamma}{r}\right)} \;=\; \int\limits_{\frac{\gamma}{t}}^{\infty} \frac{\mathrm{d}\tau}{\tau\Phi^{-1}(\tau)}, \qquad t \in (0,1]\,,$$

where $\gamma/t \geq \gamma \geq 1 > \Phi(+0) = 0$. Hence, by the condition (7.5.5) and Proposition 2.3

$$I(t) \;\leq\; I(1) \;=\; \int\limits_0^1 \frac{\mathrm{d}r}{rK(r)} \;<\; \infty. \tag{7.5.9}$$

Next, consider the mapping

$$f(z) \;=\; \frac{z}{|z|}\,\rho(|z|),$$

where $\rho(t) = \mathrm{e}^{I(t)}$. Note that $f \in C^1(\mathbb{D}\setminus\{0\})$, and hence, f is locally quasiconformal in the punctured unit disk $\mathbb{D}\setminus\{0\}$ by the continuity of the function $K(r)$, $r \in (0,1)$; see also (7.5.8). Let us calculate its complex dilatation. Set $z = r\mathrm{e}^{\mathrm{i}\vartheta}$. Then

$$\frac{\partial f}{\partial r} \;=\; \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \;+\; \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial r} \;=\; \mathrm{e}^{\mathrm{i}\vartheta}\cdot\frac{\partial f}{\partial z} \;+\; \mathrm{e}^{-\mathrm{i}\vartheta}\cdot\frac{\partial f}{\partial \overline{z}}$$

and

$$\frac{\partial f}{\partial \vartheta} \;=\; \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} \;+\; \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \vartheta} \;=\; \mathrm{i}r\mathrm{e}^{\mathrm{i}\vartheta}\cdot\frac{\partial f}{\partial z} \;-\; \mathrm{i}r\mathrm{e}^{-\mathrm{i}\vartheta}\cdot\frac{\partial f}{\partial \overline{z}}\,.$$

In other words,

$$\frac{\partial f}{\partial z} \;=\; \frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{2}\left(\frac{\partial f}{\partial r} \;+\; \frac{1}{ir}\cdot\frac{\partial f}{\partial \vartheta}\right) \tag{7.5.10}$$

and

$$\frac{\partial f}{\partial \overline{z}} \;=\; \frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2}\left(\frac{\partial f}{\partial r} \;-\; \frac{1}{ir}\cdot\frac{\partial f}{\partial \vartheta}\right). \tag{7.5.11}$$

Thus, we have that

$$\frac{\partial f}{\partial z} \;=\; \frac{1}{2}\left(\frac{\rho(r)}{rK(r)} \;+\; \frac{\rho(r)}{r}\right) \;=\; \frac{\rho(r)}{2r}\;\cdot\frac{1+K(r)}{K(r)}$$

and

$$\frac{\partial f}{\partial \overline{z}} \;=\; \frac{\mathrm{e}^{2\mathrm{i}\vartheta}}{2}\left(\frac{\rho(r)}{rK(r)} \;-\; \frac{\rho(r)}{r}\right) \;=\; \mathrm{e}^{2\mathrm{i}\vartheta}\cdot\frac{\rho(r)}{2r}\;\cdot\frac{1-K(r)}{K(r)},$$

i.e.,

$$\mu(z) \;=\; \mathrm{e}^{2\mathrm{i}\vartheta}\cdot\frac{1-K(r)}{1+K(r)} \;=\; -\;\frac{z}{\overline{z}}\cdot\frac{K(|z|)-1}{K(|z|)+1}\,.$$

Consequently

$$K_\mu(z) \;=\; K(|z|), \tag{7.5.12}$$

and by (7.5.7)

$$\int_{\mathbb{D}} \Phi(K_\mu(z))\, \mathrm{d}m(z) \;=\; 2\pi \int_0^1 \Phi(K(r))\, r\, \mathrm{d}r \;\leq\; 2\pi\gamma^2 I(1) \;<\; \infty.$$

However,

$$\lim_{z\to 0} |f(z)| \;=\; \lim_{t\to 0} \rho(t) \;=\; \mathrm{e}^{I(0)} \;=\; 1,$$

i.e., f maps the punctured disk $\mathbb{D}\setminus\{0\}$ onto the ring $1<|\zeta|<R=\mathrm{e}^{I(1)}$.

Let us assume that there is a homeomorphic ACL solution g of the Beltrami equation (B) with the given μ. By the Riemann theorem, without loss of generality, we may assume that $g(0)=0$ and $g(\mathbb{D})=\mathbb{D}$. Since both f and g are locally quasiconformal in the punctured disk $\mathbb{D}\setminus\{0\}$, then by the uniqueness theorem for the quasiconformal mappings, $f=h\circ g$ in $\mathbb{D}\setminus\{0\}$ where h is a conformal mapping in $\mathbb{D}\setminus\{0\}$. However, isolated singularities are removable for conformal mappings. Hence, h can be extended by continuity to 0 and, consequently, f should be so. Thus, the obtained contradiction disproves the assumption (7.5.5). □

Remark 7.7. Theorems 2.3 and 7.8 show that each of the conditions (2.8.12)–(2.8.17) in the existence theorems for the Beltrami equations (B) with the integral constraints (7.5.1) for nondecreasing convex functions Φ is not only sufficient, but also necessary.

7.6 Representation, Factorization, and Uniqueness Theorems

The following proposition is easily obtained by Stoilow's factorization theorem; see, e.g., [240], and also Proposition 2.4.

Proposition 7.1. *Let $\mu : D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. such that*

$$K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;\in\; L^1_{\mathrm{loc}}. \tag{7.6.1}$$

Then every (continuous) discrete and open ACL *solution g of the Beltrami equation (B) has the representation*

$$g=h\circ f, \tag{7.6.2}$$

where f is a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of (B) and h is a holomorphic function in $f(D)$.

Iwaniec and Sverák [118] showed that, if $K_\mu\in L^1_{\mathrm{loc}}$, then every $\mathrm{W}^{1,2}_{\mathrm{loc}}$ solution g of (B) has the representation $g=h\circ f$ for some holomorphic function h and some homeomorphism f. Iwaniec and Martin have constructed ACL solutions for the Beltrami equation which are not in $W^{1,2}_{\mathrm{loc}}$ and not open and discrete and, thus, have no such representation (7.6.2); see, e.g., [116].

Remark 7.8. As a consequence of Proposition 7.1, we obtain that if $K_\mu \in L^1_{\mathrm{loc}}$, then either the Beltrami equation (B) has a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution or has no continuous, discrete, and open ACL solution. Note that, for every $p \in [1,\infty)$, there are examples of measurable functions $\mu : \mathbb{C} \to \mathbb{C}$ such that $|\mu(z)| < 1$ a.e. and $K_\mu(z) \in L^p_{\mathrm{loc}}$, and for which the Beltrami equation (B) has no homeomorphic ACL solution; see Proposition 5.4.

Another type of condition for the uniqueness of a homeomorphic ACL solution can be obtained by imposing some conditions on the "size" of the singular set of μ. This will be done in Lemma 7.3 and Theorem 7.9 below.

Recall that the *singular set* S_μ of $\mu : D \to \mathbb{C}$ is defined by

$$S_\mu = \left\{ z \in D : \lim_{\varepsilon\to 0} \|K_\mu\|_{L^\infty(B(z,\varepsilon))} \;=\; \infty \right\}. \tag{7.6.3}$$

Obviously, the set S_μ is closed relatively to the domain D.

Now, let (X,d) be a metric space and $H = \{h_x(r)\}_{x\in X}$ a family of functions $h_x : (0,\rho_x) \to (0,\infty)$, $\rho_x > 0$, such that $h_x(r) \to 0$ as $r \to 0$. Set

$$L^\rho_H(X) \;=\; \inf\, \Sigma\, h_{x_k}(r_k), \tag{7.6.4}$$

where the infimum is taken over all finite collections of $x_k \in X$ and $r_k \in (0,\rho)$ such that the balls

$$B(x_k,r_k) = \{x \in X : d(x,x_k) < r_k\} \tag{7.6.5}$$

cover X. The limit

$$L_H(X) \;:\;=\; \lim_{\rho\to 0} L^\rho_H(X) \tag{7.6.6}$$

exists. We call $L_H(X)$ the *H-length* of X. In the particular case where $h_x(r) = r$ for all $x \in X$ and $r > 0$, the H-length is the usual (Hausdorff) length of X.

Lemma 7.3. *Let* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and* $K_\mu \in L^1_{\mathrm{loc}}$. *Suppose that for every* $z_0 \in D$ *there exist* $\varepsilon_0 = \delta(z_0) < \operatorname{dist}(z_0,\partial D)$ *and a one-parameter family of measurable functions* $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, *such that*

$$0 \;<\; I_{z_0}(\varepsilon) \colon \;=\; \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,\mathrm{d}t \;<\; \infty, \quad \varepsilon \in (0,\varepsilon_0), \tag{7.6.7}$$

and such that

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\,\mathrm{d}m(z) \;=\; o\left(I^2_{z_0}(\varepsilon)\right) \tag{7.6.8}$$

as $\varepsilon \to 0$. *Let* f_μ *be a strong ring solution of (B).*

If the singular set S_μ *is of H-length zero for* $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$ *with*

$$h_{z_0}(r) \ = \exp\left(-\frac{2\pi}{\omega_{z_0}(r)}\right), \quad z_0 \in S_\mu, \quad r \in (0, \delta(z_0)), \tag{7.6.9}$$

where

$$\omega_{z_0}(\varepsilon) \ = \ \frac{1}{I^2_{z_0}(\varepsilon)} \int\limits_{A(\varepsilon)} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|) \ \mathrm{d}m(z), \tag{7.6.10}$$

then every homeomorphic ACL solution g of (B) has the representation $g = h \circ f_\mu$ *for some conformal mapping h in* $f_\mu(D)$.

Proof. If $L_H(S_\mu) = 0$, then $S'_\mu = f_\mu(S_\mu)$ is of length zero by Corollary 6.6. Consequently, S'_μ does not locally disconnect $f_\mu(D)$ (see, e.g., [259]), and hence, $G = D \setminus S_\mu$ is a domain. The homeomorphisms g and f_μ are locally quasiconformal in the domain G and hence $h = g \circ f_\mu^{-1}$ is conformal in the domain $f_\mu(D) \setminus S'_\mu$. Since S'_μ is of length zero, it is removable for h, i.e., h can be extended to a conformal mapping in $f_\mu(D)$ by the Painleve theorem; see, e.g., [40] or [68]. □

Theorem 7.9. *Let* $\mu : D \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and* $K_\mu \in L^1_{\mathrm{loc}}$. *Suppose that every point* $z_0 \in D$ *has a neighborhood* $U_{z_0} = B(z_o, \delta(z_0))$ *for some* $0 < \delta(z_0) < \mathrm{dist}\,(z_0, \partial D)$ *and a measurable function* $Q_{z_0}(z) : U_{z_0} \to [0, \infty]$ *such that*

$$K^T_\mu(z, z_0) \ \leq \ Q_{z_0}(z) \quad \text{a.e.in } U_{z_0} \tag{7.6.11}$$

and

$$\int\limits_0^{\delta(z_0)} \frac{\mathrm{d}t}{t q_{z_0}(t)} \ = \ \infty, \tag{7.6.12}$$

where $q_{z_0}(t)$ *is the mean value of* $Q_{z_0}(z)$ *over the circle* $|z - z_0| = t$. *Let* f_μ *be a strong ring solution of (B).*

If the singular set S_μ *has H-length zero for* $H = \{h_{z_0}(r)\}_{z_0 \in S_\mu}$ *where*

$$h_{z_0}(r) \ = \ \exp\left(-\int\limits_r^{\delta(z_0)} \frac{\mathrm{d}t}{t q_{z_0}(t)}\right), \quad z_0 \in S_\mu, \quad r \in (0, \delta(z_0)), \tag{7.6.13}$$

then every homeomorphic ACL solution g of (B) has the representation $g = h \circ f_\mu$ *for some conformal mapping h in* $f_\mu(D)$.

Proof. Theorem 7.9 follows from Lemma 7.3 with

$$\psi_{z_0,\varepsilon}(t) \ \equiv \ \psi_{z_0}(t) \ : \ = \begin{cases} 1/[t q_{z_0}(t)], & t \in (0, \varepsilon_0), \\ 0, & \text{otherwise}, \end{cases} \tag{7.6.14}$$

where $\varepsilon_0 = \delta(z_0)$ because

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi_{z_0}^2(|z-z_0|)\,\mathrm{d}m(z) \;=\; 2\pi\int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0}(t)\,\mathrm{d}t. \tag{7.6.15}$$

□

Remark 7.9. Let the functions $Q_{z_0}(z)$ satisfies the condition

$$M_{z_0} \;:\;=\; \fint_{U_{z_0}} \Phi(Q_{z_0}(z))\,\mathrm{d}m(z) \;<\; \infty \tag{7.6.16}$$

and the singular set S_μ be of H-length zero for $H = \{h_{z_0}(r)\}_{z_0\in S_\mu}$ with

$$h_{z_0}(r) \;:\;=\; \exp\left(-\frac{1}{2}\int\limits_{eM_{z_0}}^{\frac{\delta(z_0)}{r}}\frac{\mathrm{d}\tau}{\tau\Phi_{z_0}^{-1}(\tau)}\right), \quad z_0\in S_\mu,\quad r\in(0,\delta(z_0)). \tag{7.6.17}$$

Here, $\Phi_{z_0}(\tau)$ stands for convex increasing functions satisfying one of the conditions (2.8.12)–(2.8.17). Then by Lemma 6.2, every homeomorphic ACL solution g of the Beltrami equation (B) has the Stoilov representation $g = h\circ f_\mu$ with a conformal mapping h in $f_\mu(D)$.

Corollary 7.14. *Let $\mu : D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. and $K_\mu\in L^1_{\mathrm{loc}}$. Suppose that every point $z_0\in D$ has a neighborhood U_{z_0} where (7.6.11) holds with a function $Q_{z_0}(z)$ of finite mean oscillation at z_0 in the variable z. Suppose also that the singular set of S_μ is of H-length zero for $H=\{h_{z_0}(r)\}_{z_0\in S_\mu}$,*

$$h_{z_0}(r) \;=\; \left(\log\frac{\delta(z_0)}{r}\right)^{-\beta(z_0)}, \quad z_0\in S_\mu,\quad r\in(0,\delta(z_0)), \tag{7.6.18}$$

where $\delta(z_0) <$ dist $(z_0,\partial D)$ and $2\beta(z_0) = (q(z_0)+6d(z_0))^{-1}$, $q(z_0)$ is the mean value of $Q_{z_0}(z)$ over $D(z_0,\delta(z_0)/2)$ and $d(z_0)$ is the maximal dispersion of $Q_{z_0}(z)$ in $D(z_0,\delta(z_0)/2)$. Let f_μ be a strong ring solution of (B).

Then every homeomorphic ACL *solution g of (B) has the representation $g = h\circ f_\mu$ for some conformal mapping h in $f_\mu(D)$.*

Corollary 7.14 follows immediately from Lemmas 5.3 and 7.3.

Remark 7.10. In view of Remark 2.2, if the condition

$$Q^*(z_0) \;:\;=\; \overline{\lim_{\varepsilon\to 0}}\; \fint_{D(z_0,\varepsilon)} Q_{z_0}(z)\,\mathrm{d}m(z) \;<\; \infty \tag{7.6.19}$$

holds for all $z_0\in D$, then one may take $\beta(z)=\gamma/Q^*(z)$ in (7.6.18) for any $\gamma<1/26$.

On the basis of Lemma 7.3, for every existence theorem in the previous sections, one can formulate the corresponding uniqueness theorem in the spirit of Theorem 7.9.

7.7 Historical Comments and Final Remarks

To compare our results with earlier results of other authors, we give a short survey.

The first investigation of the existence problem for degenerate Beltrami equations with integral constraints (7.4.9) as in Theorem 7.7 was conducted by Pesin [185], who studied the special case where $\Phi(t) = e^{t^\alpha} - 1$ with $\alpha > 1$. Basically, Corollary 7.13 is due to Kruglikov [133]. David [70] considered the existence problem with measure constraints

$$|\{z \in D :\ K_\mu(z) \ > \ t\}| \ \leq \ \varphi(t) \qquad \forall\, t \in [1,\infty) \tag{7.7.1}$$

with special $\varphi(t)$ of the form $a \cdot e^{-bt}$ and Tukia [254] with the corresponding constraints in terms of the spherical area. Note that under the integral constraints (7.4.9) of the exponential type $\Phi(t) = \alpha e^{\beta t}$, $\alpha > 0$, the conditions of David and Tukia hold. Thus, the latter results strengthen the Pesin result.

By the well-known John–Nirenberg lemma for functions of the class BMO, the David conditions are equivalent to the corresponding integral conditions of exponential type; see, e.g., [212]. More advanced results in terms of FMO can be found in [217] and [220].

The next step was made by Brakalova and Jenkins [56], who proved the existence of ACL homeomorphic solutions for the case of the integral constraints (7.4.1), as in Theorem 7.6 with $K_\mu(z)$ instead of $K^T_\mu(z,z_0)$ and with

$$\Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ \exp\left(\frac{\frac{t+1}{2}}{1+\log\frac{t+1}{2}}\right). \tag{7.7.2}$$

Note that, for the case in [56], the condition (2.8.12) in Theorem 2.3 (see also Corollary 7.8) can be easily verified by the calculations

$$(\log \Phi(t))' \ = \ \frac{1}{2}\,\frac{\log\frac{t+1}{2}}{(1+\log\frac{t+1}{2})^2} \ \sim \ \frac{1}{2}\,\frac{1}{\log t} \qquad \text{as } t \to \ \infty. \tag{7.7.3}$$

Moreover, it is easy to verify that $\Phi''(t) \geq 0$ for all $t \geq T$ under large enough $T \in (1,\infty)$, and thus, Φ is convex on the segment $[T,\infty]$; see, e.g., [53] and Remark 7.5.

Later on, Iwaniec and Martin proved the existence of solutions in the Orlicz–Sobolev classes for the case where

$$\Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ \exp\left(\frac{pt}{1+\log t}\right) \tag{7.7.4}$$

for some $p > 0$ (see, e.g., [116, 117]) for which

$$(\log \Phi(t))' = \frac{p \log t}{(1+\log t)^2} \sim \frac{p}{\log t} \qquad \text{as } t \to \infty; \tag{7.7.5}$$

cf. Corollary 7.11. Note that in both cases (7.7.2) and (7.7.4)

$$\Phi(t) \geq t^l \qquad \forall\, t \geq t_l \in [1,\infty). \tag{7.7.6}$$

It is splendid that in the case of

$$\Phi_{z_0}(t) \equiv \Phi(t) = \exp\, pt, \tag{7.7.7}$$

uniqueness and factorization theorems for solutions of the Beltrami equations of the Stoilow type were established; see, e.g., [26] and [70].

Corollary 7.7 is due to Gutlyanskii, Martio, Sugawa, and Vuorinen in [98] and [99], where they have established the existence of ACL homeomorphic solutions of the Beltrami equation (B) in $W^{1,s}_{\mathrm{loc}}$, $s = 2p/(1+p)$, under $K_\mu \in L^p_{\mathrm{loc}}$ with $p > 1$ for

$$\Phi_{z_0}(t) \equiv \Phi(t) := \exp H(t), \tag{7.7.8}$$

with $H(t)$ being a continuous nondecreasing function such that $\Phi(t)$ is convex and

$$\int_1^\infty H(t)\,\frac{dt}{t^2} = \infty. \tag{7.7.9}$$

It was one of the most outstanding results in the field of criteria for the solvability of the degenerate Beltrami equations, as it is clear from Theorem 7.8; see also Remark 7.7.

Subsequently, the fine theorems on the existence and uniqueness of solutions in the Orlich–Sobolev classes were established under condition (7.7.9) with smooth H and the condition $tH'(t) \geq 5$; see Theorem 20.5.2 in the monograph [26], cf. Lemma 7.2, see also Remark 7.6 above. However, we have not found [99] in the reference list of this monograph. The theorems on the existence and uniqueness of solutions in the class $W^{1,2}_{\mathrm{loc}}$ were also established earlier for $K_\mu(z) \leq Q(z) \in W^{1,2}_{\mathrm{loc}}$ in [158].

Recently, Brakalova and Jenkins have proved the existence of ACL homeomorphic solutions under (7.4.1), again with $K_\mu(z)$ instead of $K^T_\mu(z,z_0)$, and with

$$\Phi_{z_0}(t) \equiv \Phi(t) = h\left(\frac{t+1}{2}\right), \tag{7.7.10}$$

where they assumed that h is increasing and convex and $h(x) \geq C_l x^l$ for any $l > 1$ with some $C_l > 0$ and

$$\int_1^\infty \frac{\mathrm{d}\tau}{\tau h^{-1}(\tau)} = \infty; \tag{7.7.11}$$

see [57]. Note that the conditions $h(x) \geq C_l x^l$ for any $l > 1$, in particular, under the above subexponential integral constraints (see (7.7.6)) imply that K_μ is locally integrable with any degree $p \in [1, \infty)$; see Proposition 2.4.

Some of the given conditions are not necessary, as is clear from the results in Sect. 7.4 and from the following lemma and remarks:

Lemma 7.4. *There exist continuous increasing convex functions $\Phi : [1,\infty) \to [1,\infty)$ such that*

$$\int_1^\infty \log \Phi(t) \, \frac{\mathrm{d}t}{t^2} = \infty, \tag{7.7.12}$$

$$\liminf_{t\to\infty} \frac{\log \Phi(t)}{\log t} = 1 \tag{7.7.13}$$

and, moreover,

$$\Phi(t) \geq t \qquad \forall\, t \in [1,\infty). \tag{7.7.14}$$

Note that the examples from the proof of Lemma 7.4 below can be extended to $[0,\infty]$ by $\Phi(t) = t$ for $t \in [0,1]$ retaining all the given properties.

Remark 7.11. The condition (7.7.13) implies, in particular, that there exist no $\lambda > 1, C_l > 0$ and $T_l \in [1,\infty)$ such that

$$\Phi(t) \geq C_l \cdot t^\lambda \qquad \forall\, t \geq T_l . \tag{7.7.15}$$

Thus, in view of Lemma 7.4 and Theorem 7.7, none of the conditions (7.7.15) is necessary in the existence theorems for the Beltrami equations with integral constraints of the type (7.4.9).

In addition, for the examples of Φ given in the proof of Lemma 7.4,

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{\log t} = \infty; \tag{7.7.16}$$

cf. Proposition 7.2 below. Finally, all the conditions (2.8.12)–(2.8.17) from Theorem 2.3 hold simultaneously with (7.7.12) because the increasing convex function Φ is absolutely continuous.

Proof. We use a known criterion which says that a function Φ is convex on an open interval I if and only if Φ is continuous and its derivative Φ' exists and is nondecreasing in I except for a countable set of points in I; see, e.g., Proposition 1.4.8

in [53]. We construct Φ by induction, sewing together pairs of functions of the two types $\varphi(t) = \alpha + \beta t$ and $\psi(t) = a\mathrm{e}^{bt}$ with suitable positive parameters a, b, and β and possibly negative α.

More precisely, set $\Phi(t) = \varphi_1(t)$ for $t \in [1, \gamma_1^*]$ and $\Phi(t) = \psi_1(t)$ for $t \in [\gamma_1^*, \gamma_1]$, where $\varphi_1(t) = t$, $\gamma_1^* = e$, $\psi_1(t) = \mathrm{e}^{-(e-1)}\mathrm{e}^t$, $\gamma_1 = e+1$. Let us assume that we have already constructed $\Phi(t)$ on the segment $[1, \gamma_n]$ and hence that $\Phi(t) = a_n \mathrm{e}^{b_n t}$ on the last subsegment $[\gamma_n^*, \gamma_n]$ of the segment $[\gamma_{n-1}, \gamma_n]$. Then we set $\varphi_{n+1}(t) = \alpha_{n+1} + \beta_{n+1} t$, where the parameters α_{n+1} and β_{n+1} are found from the conditions $\varphi_{n+1}(\gamma_n) = \Phi(\gamma_n)$ and $\varphi'_{n+1}(\gamma_n) \geq \Phi'(\gamma_n - 0)$, i.e., $\alpha_{n+1} + \beta_{n+1}\gamma_n = a_n \mathrm{e}^{b_n \gamma_n}$ and $\beta_{n+1} \geq a_n b_n \mathrm{e}^{b_n \gamma_n}$. Let $\beta_{n+1} = a_n b_n \mathrm{e}^{b_n \gamma_n}$, $\alpha_{n+1} = a_n \mathrm{e}^{b_n \gamma_n}(1 - b_n \gamma_n)$, choose a large enough $\gamma_{n+1}^* > \gamma_n$ from the condition

$$\log\left(\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*\right) \leq \left(1 + \frac{1}{n}\right) \log \gamma_{n+1}^* \tag{7.7.17}$$

and, finally, set $\Phi(t) \equiv \varphi_{n+1}(t)$ on $[\gamma_n, \gamma_{n+1}^*]$.

Next, we set $\psi_{n+1}(t) = a_{n+1}\mathrm{e}^{b_{n+1}t}$, where parameters a_{n+1} and b_{n+1} are found from the conditions that $\psi_{n+1}(\gamma_{n+1}^*) = \varphi_{n+1}(\gamma_{n+1}^*)$ and $\psi'_{n+1}(\gamma_{n+1}^*) \geq \varphi'_{n+1}(\gamma_{n+1}^*)$, i.e.,

$$b_{n+1} = \frac{1}{\gamma_{n+1}^*} \log \frac{\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*}{a_{n+1}} \tag{7.7.18}$$

and, taking into account (7.7.18),

$$b_{n+1} \geq \frac{\beta_{n+1}}{\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*}. \tag{7.7.19}$$

Note that (7.7.19) holds if we take small enough $a_{n+1} > 0$ in (7.7.18). In addition, we may choose here $b_{n+1} > 1$.

Now, let us choose a large enough γ_{n+1} with $\mathrm{e}^{-1}\gamma_{n+1} \geq \gamma_{n+1}^*$ from the condition that

$$\log \psi_{n+1}\left(\mathrm{e}^{-1}\gamma_{n+1}\right) \geq \mathrm{e}^{-1}\gamma_{n+1}, \tag{7.7.20}$$

i.e.,

$$\log a_{n+1} + b_{n+1}\mathrm{e}^{-1}\gamma_{n+1} \geq \mathrm{e}^{-1}\gamma_{n+1}. \tag{7.7.21}$$

Note that (7.7.21) holds for all large enough γ_{n+1} because $b_{n+1} > 1$, although $\log a_{n+1}$ can be negative.

Setting $\Phi(t) = \psi_{n+1}(t)$ on the segment $[\gamma_{n+1}^*, \gamma_{n+1}]$, we have that

$$\log \Phi(t) \geq t \qquad \forall t \in [\mathrm{e}^{-1}\gamma_{n+1}, \gamma_{n+1}], \tag{7.7.22}$$

where the subsegment $[\mathrm{e}^{-1}\gamma_{n+1}, \gamma_{n+1}] \subseteq [\gamma_{n+1}^*, \gamma_{n+1}]$ has logarithmic length 1.

Thus, (7.7.14) holds because by construction, $\Phi(t)$ is absolutely continuous, $\Phi(1)=1$ and $\Phi'(t)\geq 1$ for all $t\in[1,\infty)$; the equality (7.7.12) holds by (7.7.22); (7.7.13) by (7.7.14) and (7.7.17); (7.7.16) by (7.7.22). □

Remark 7.12. If we take $\beta_{n+1}=1$ for all $n=1,2,\ldots$, $\alpha_{n+1}=b_n e^{b_n\gamma_n}-\gamma_n$ and arbitrary $\gamma^*_{n+1}>\gamma_{n+1}$ in the above construction in Lemma 7.4, we obtain examples of absolutely continuous increasing functions Φ which are not convex but satisfy (7.7.12), as well as all the conditions (2.8.12)–(2.8.17) from Proposition 2.3, and (7.7.14).

The corresponding examples of nondecreasing functions Φ which are neither continuous, nor strictly monotone, nor convex in any neighborhood of ∞, but satisfy (7.7.12), as well as (2.8.12)–(2.8.17), and (7.7.14), are obtained in the above construction if we take $\beta_{n+1}=0$ and $\alpha_{n+1}>\gamma_n$ such that $\alpha_{n+1}>\Phi(\gamma_n)$ and $\Phi(t)=\alpha_{n+1}$ for all $t\in(\gamma_n,\gamma^*_{n+1}]$, $\gamma^*_{n+1}=\alpha_{n+1}$.

Proposition 7.2. *Let $\Phi:[1,\infty)\to[1,\infty)$ be a locally integrable function such that*

$$\int_1^\infty \log\Phi(t)\,\frac{dt}{t^2} = \infty. \tag{7.7.23}$$

Then

$$\limsup_{t\to\infty}\frac{\Phi(t)}{t^\lambda} = \infty \qquad \forall\,\lambda\in\mathbb{R}. \tag{7.7.24}$$

Remark 7.13. In particular, (7.7.24) itself implies the relation (7.7.16). Indeed, we have from (7.7.24) that there exists a monotone sequence $t_n\to\infty$ as $n\to\infty$ such that

$$\Phi(t_n)\geq t_n^n \qquad n=1,2,\ldots, \tag{7.7.25}$$

i.e.,

$$\frac{\log\Phi(t_n)}{\log t_n}\geq n \qquad n=1,2,\ldots. \tag{7.7.26}$$

Proof. It is sufficient to consider the case $\lambda>0$. Set $H(t)=\log\Phi(t)$, i.e., $\Phi(t)=e^{H(t)}$. Note that $e^x\geq x^n/n!$ for all $x\geq 0$ and $n=1,2\ldots$ because $e^x=\sum_{n=0}^\infty x^n/n!$. Fix $\lambda>0$ and $n>\lambda$. Then $q:=\lambda/n$ belongs to $(0,1)$ and

$$\frac{H(t)}{t^q}\leq\left(\frac{\Phi(t)}{t^\lambda}\right)^{\frac{1}{n}}\cdot\sqrt[n]{n!}\,.$$

Let us assume that

$$C:=\limsup_{t\to\infty}\frac{\Phi(t)}{t^\lambda}<\infty. \tag{7.7.27}$$

Then

$$\int_{\triangle}^{\infty} H(t)\,\frac{\mathrm{d}t}{t^2} \;<\; 2\sqrt[n]{Cn!}\int_{\triangle}^{\infty}\frac{\mathrm{d}t}{t^{2-q}} \;=\; -\,\frac{2}{1-q}\,\frac{\sqrt[n]{Cn!}}{t^{1-q}}\,\Big|_{\triangle}^{\infty}$$

$$=\;\frac{2}{1-q}\,\frac{\sqrt[n]{Cn!}}{\triangle^{1-q}}\;<\;\infty$$

for large enough $\triangle > 1 > 0$. The latter contradicts (7.7.23). Hence, the assumption (7.7.27) was not true, and thus, (7.7.24) holds for all $\lambda \in \mathbb{R}$. □

Remark 7.14. Lemma 7.4 shows that, generally speaking, lim sup in (7.7.24) cannot be replaced by lim for an arbitrary $\lambda > 1$ under the condition (7.7.23) even if Φ is continuous, increasing and convex.

Chapter 8
On the Dirichlet Problem for Beltrami Equations

8.1 Introduction

Boundary problems for the Beltrami equation (B) are due to the famous dissertation of Riemann who considered a particular case of analytic functions when $\mu(z) \equiv 0$ and to the works of Hilbert (1904, 1924) who studied the corresponding system of Cauchy–Riemann for the real and imaginary parts of analytic functions $f = u + \mathrm{i}v$, as well as to the work of Poincaré (1910) on rising tides.

For further commentary on the history of the question, see Chap. 4 of the monograph [263] where the Dirichlet problem was studied for the uniformly elliptic Beltrami systems; see also the work [71]. Recent results on the existence of regular homeomorphic solutions for degenerate Beltrami equations (see, e.g., in [49,99,105, 117, 159, 165, 215, 219, 220]) and the development of the theory of the boundary behavior of regular homeomorphisms make it possible to proceed further in the domain of the existence of regular solutions of the Dirichlet problem; see the two previous chapters of the present monograph. Here, we mainly follow the original papers [76] and [77].

The concept of a ring Q-homeomorphism at inner points was first introduced and used to study degenerate Beltrami equations (B), in [215], and at boundary points in [222]. The papers [215] and [219] contained a series of theorems on the existence of solutions of (B) which are ring Q-homeomorphisms at every point $z_0 \in D$ possibly which $Q(z) < 1$ on a set of a positive measure.

The problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$, first in the case $Q \in \mathrm{BMO}$ (bounded mean oscillation) in the papers [163–165], and then for the case $Q \in \mathrm{FMO}$ (finite mean oscillation) and other cases in the papers [113, 114, 207]; see also the monograph [165]. In the paper [225], ACL and differentiability a.e. for Q-homeomorphisms in $\mathbb{R}^n$, $n \geq 2$, was established whenever $Q \in L^1_{\mathrm{loc}}$. Moreover, it was shown there that such Q-homeomorphisms f belong to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and $J_f(z) \neq 0$ a.e.

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8.2 Characterization of Ring Q-Homeomorphisms at the Boundary

Below, we use the standard conventions: $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$; see, e.g., [224]. Set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Lemma 8.1. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function. A homeomorphism $f : \mathbb{D} \to \mathbb{D}$ is a ring Q-homeomorphism at a point $z_0 \in \partial\mathbb{D}$ if and only if*

$$M\left(\Delta\left(fC_1^*, fC_2^*, f\left(A \cap \mathbb{D}\right)\right)\right) \le \int\limits_{A\cap\mathbb{D}} Q(z) \cdot \eta^2(|z - z_0|)\, \mathrm{d}m(z) \tag{8.2.1}$$

for every ring $A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, $0 < r_1 < r < r_2 < 2$, where $C_1^* = S(z_0, r_1) \cap \mathbb{D}$ and $C_2^* = S(z_0, r_2) \cap \mathbb{D}$ are boundaries of the ring $A = A(z_0, r_1, r_2)$ in $\mathbb{D}$, and for every measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, \mathrm{d}r \ge 1. \tag{8.2.2}$$

Proof. Indeed, the family of paths $\Delta(fC_1^*, fC_1^*, f\,(A \cap \mathbb{D}))$ minorizes every family $\Delta(fC_1, fC_1, \mathbb{D})$, where $C_1$and C_2 are arbitrary continua in $\mathbb{D}$ belonging to the different components of the complement of the ring A, and thus

$$M(\Delta(fC_1, fC_2, \mathbb{D})) \le M(\Delta(fC_1^*, fC_2^*, f(A \cap \mathbb{D})));$$

see, e.g., 6.4 in [258]. Consequently, (8.2.1) implies that

$$M\left(\Delta\left(fC_1, fC_2, fG\right)\right) \le \int\limits_{A\cap\mathbb{D}} Q(z) \cdot \eta^2(|z - z_0|)\, \mathrm{d}m(z)\,. \tag{8.2.3}$$

Let us show that (8.2.1) holds for every ring Q-homeomorphism $f : \mathbb{D} \to \mathbb{D}$ at the point z_0. For this goal, take in (8.2.3) expanding sequences $C_1 = C_1^n$ and $C_2 = C_2^n$ of closed arcs which exhaust the open arcs C_1^* and C_2^*, respectively. Then $\Delta(fC_1^*, fC_2^*, \mathbb{D}) = \bigcup_{n=1}^{\infty} \Delta(fC_1^n, fC_2^n, \mathbb{D})$ and, consequently (see, e.g., [272]),

$$M(\Delta(fC_1^*, fC_2^*, \mathbb{D})) = \lim_{n\to\infty} M(\Delta(fC_1^n, fC_2^n, \mathbb{D}))\,.$$

Thus, (8.2.3) also implies (8.2.1) because

$$\Delta(fC_1^*, fC_2^*, f\,(A \cap \mathbb{D})) \subseteq \Delta(fC_1^*, fC_2^*, \mathbb{D})\,. \qquad \square$$

Lemma 8.2. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function and let $f : \mathbb{D} \to \mathbb{D}$ be a ring Q-homeomorphism at a point $z_0 \in \partial\mathbb{D}$. Then*

$$M\left(\Delta\left(fC_1^*, fC_2^*, f\left(A\cap\mathbb{D}\right)\right)\right) \leq \left(\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)}\right)^{-1} \tag{8.2.4}$$

for every $0 < r_1 < r_2 < 2$, *where* $\|Q\|_1(r) = \int_{\gamma_r} Q(z)|\mathrm{d}z|$ *is the norm in* L^1 *of the function* Q *over the arcs* $\gamma_r = \mathbb{D}\cap S(z_0,r)$, $r \in (r_1,r_2)$, $C_1^* = S(z_0,r_1)\cap\mathbb{D}$ *and* $C_2^* = S(z_0,r_2)\cap\mathbb{D}$, $A = \{z\in\mathbb{C} : r_1 < |z-z_0| < r_2\}$.

Proof. Set

$$I = \int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)}.$$

Without loss of generality, we may assume that $I \neq 0$ because otherwise (8.2.4) is trivial and that $I \neq \infty$ because otherwise in (8.2.4) we can replace $Q_\delta(z)$ by $Q(z)+\delta$ with an arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (8.2.4).

Let $I \neq \infty$. Then $\|Q\|_1(r) \neq 0$ a.e. on (r_1, r_2). Set

$$\psi(t) = \begin{cases} \frac{1}{\int\limits_{\gamma_t} Q(z)|\mathrm{d}z|}, & t \in (r_1,r_2), \\ 0, & t \notin (r_1,r_2). \end{cases}$$

Then, by the Fubini theorem applied to the polar coordinates centered at the point z_0, we have

$$\int\limits_{A\cap\mathbb{D}} Q(z)\cdot\psi^2(|z-z_0|)\,\mathrm{d}m(z) = \int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\int\limits_{\gamma_r} Q(z)\,|\mathrm{d}z|}. \tag{8.2.5}$$

Let Γ be the family of all paths joining the circles C_1^* and C_2^* in $A\cap\mathbb{D}$. Let also ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in [0,\infty]$. Such a function ψ^* exists by the Lusin theorem (see, e.g., 2.3.5 in [83] and [224], p. 72). Then the function

$$\rho(z) = \psi^*(|z-z_0|)/I$$

is admissible for the family Γ and, according to relation (8.2.5), for every ring Q-homeomorphism at the point z_0, we have that

$$M(f\Gamma) \leq \int\limits_{A\cap\mathbb{D}} Q(z)\cdot\rho^2(z)\,\mathrm{d}m(z) = \left(\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\int\limits_{\gamma_r} Q(z)\,|\mathrm{d}z|}\right)^{-1}. \qquad \square$$

The following lemma shows that inequality (8.2.4), generally speaking, cannot be improved for ring Q-homeomorphisms:

Lemma 8.3. *Let* $Q : \mathbb{D} \to [0,\infty]$ *be a measurable function and let* $z_0 \in \partial\mathbb{D}$. *Set*

$$\eta_0(r) = \frac{1}{\|Q\|_1(r)\int\limits_{r_1}^{r_2} \frac{\mathrm{d}t}{\|Q\|_1(t)}},$$

where $\|Q\|_1(r) = \int\limits_{\gamma_r} Q(z)|\mathrm{d}z|$ *and* $\gamma_r = \mathbb{D} \cap S(z_0, r)$. *Then*

$$\left(\int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)}\right)^{-1} = \int\limits_{A\cap\mathbb{D}} Q(z)\cdot\eta_0^2(|z-z_0|)\,\mathrm{d}m(z) \le \int\limits_{A\cap\mathbb{D}} Q(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z) \tag{8.2.6}$$

for every function $\eta : (r_1, r_2) \to [0,\infty]$ *such that*

$$\int\limits_{r_1}^{r_2} \eta(r)\,\mathrm{d}r = 1\,. \tag{8.2.7}$$

Proof. Set

$$I = \int\limits_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)}.$$

If $I = \infty$, then the left-hand side in (8.2.6) is equal to zero and the inequality is obvious. If $I = 0$, then $\|Q\|_1(r) = \infty$ for a.e. $r \in (r_1, r_2)$, and both sides in (8.2.6) are equal to ∞. Hence, we may assume further that $0 < I < \infty$. Then it follows from (8.2.6) that $\|Q\|_1(r) \ne 0$ a.e. in (r_1, r_2). By (8.2.7), $\eta(r) \ne \infty$ a.e. in (r_1, r_2). Setting

$$\alpha(r) = \|Q\|_1(r)\,\eta(r)\,, \quad w(r) = \frac{1}{\|Q\|_1(r)}\,,$$

we have that

$$\eta(r) = \alpha(r)w(r)$$

a.e. in (r_1, r_2). Set

$$C := \int\limits_{A\cap\mathbb{D}} Q(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z) = \int\limits_{r_1}^{r_2} \alpha^2(r)\cdot w(r)\,\mathrm{d}r.$$

By Jensen's inequality with weights $w(r)$ (see, e.g., Theorem 2.6.2 in [196]) applied to the convex function $\varphi(t) = t^2$ and to the probability measure

$$\nu(E) = \frac{1}{I}\int\limits_E w(r)\,\mathrm{d}r, \quad E \subset \Omega,$$

we obtain that

$$\left(⨍ \alpha^2(r)w(r)\,\mathrm{d}r\right)^{1/2} \ge ⨍ \alpha(r)w(r)\,\mathrm{d}r = \frac{1}{I},$$

where we have also used the fact that $\eta(r) = \alpha(r)w(r)$ satisfies relation (8.2.7) and

$$I = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)}.$$

Thus,

$$C \geq \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)} \right)^{-1},$$

and the proof is complete. □

Theorem 8.1. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function. A homeomorphism $f : \mathbb{D} \to \mathbb{D}$ is a ring Q-homeomorphism at a point $z_0 \in \partial\mathbb{D}$ if and only if*

$$M\left(\Delta\left(fC_1^*, fC_2^*, f\left(\mathbb{D} \cap A\right)\right)\right) \leq \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\|Q\|_1(r)} \right)^{-1} \tag{8.2.8}$$

for every $0 < r_1 < r_2 < 2$, where $\|Q\|_1(r) = \int_{\gamma_r} Q(z)|\mathrm{d}z|$ is the norm in L^1 of the function Q over the arcs $\gamma_r = \mathbb{D} \cap S(z_0, r)$, $r \in (r_1, r_2)$, $C_1^* = S(z_0, r_1) \cap \mathbb{D}$ and $C_2^* = S(z_0, r_2) \cap \mathbb{D}$, $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$.

Remark 8.1. Note that the infimum from the right-hand side in (8.2.3) yields the function

$$\eta_0(r) = \frac{1}{\|Q\|_1(r) \int_{r_1}^{r_2} \frac{\mathrm{d}t}{\|Q\|_1(t)}}.$$

8.3 The Regular Homeomorphisms

A homeomorphism f of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is called *regular* if its *Jacobian* $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ a.e. Every regular homeomorphism satisfies some Beltrami equation (B) with complex coefficient $\mu(z) = \mu_f(z) = f_{\bar{z}}/f_z$.

Lemma 8.4. *Let $f : \mathbb{D} \to \mathbb{D}$ be a regular homeomorphism and let $z_0 \in \partial\mathbb{D}$. Then*

$$M\left(\Delta\left(fC_1^*, fC_2^*, f\left(\mathbb{D} \cap A\right)\right)\right) \leq \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\|K_\mu^T\|_1(r)} \right)^{-1} \tag{8.3.1}$$

for $0 < r_1 < r_2 < 2$, where $\|K_\mu^T\|_1(r) = \int_{\gamma_r} K_\mu^T(z, z_0)|\mathrm{d}z|$ is the norm in L^1 of the function $K_\mu^T(z, z_0)$ over the arcs $\gamma_r = \mathbb{D} \cap S(z_0, r)$, $r \in (r_1, r_2)$, $C_1^* = S(z_0, r_1) \cap \mathbb{D}$ and $C_2^* = S(z_0, r_2) \cap \mathbb{D}$, $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$.

Proof. Set $\Gamma = \Delta\left(fC_1^*, fC_2^*, f(A\cap\mathbb{D})\right)$. Consider the quadrilateral $R = \{z\in\mathbb{D} : r_1 < |z-z_0| < r_2\}$. Then there is a conformal map h mapping fR onto a circle semiring $R' = \{w : 1<|w|<L, \operatorname{Im} w > 0\}$. Let Γ^* be the family of all paths joining the boundary components $|w|=1$ and $|w|=L$ of the semiring R'. Then, in view of the conformal invariance of the modulus, $M(\Gamma^*) = M(\Gamma)$. Thus,

$$M(\Gamma) = \frac{\pi^2}{\int\limits_{R'} \frac{\mathrm{d}u\mathrm{d}v}{|w|^2}},$$

where $w = u + \mathrm{i}v$.

For $g = h\circ f$, we have that $g \in W^{1,1}_{\mathrm{loc}}\left(f(A\cap\mathbb{D})\right)$, and hence, g is a.e. absolutely continuous and differentiable a.e. on γ_r for a.e. $r\in(r_1,r_2)$; see, e.g., 1.1.7 in [169]. Note that

$$\int\limits_{r_1}^{r_2}\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{J_g(z_0+re^{\mathrm{i}\theta})}{|g(z_0+re^{\mathrm{i}\theta})|^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta \le \int\limits_{R'}\frac{\mathrm{d}u\,\mathrm{d}v}{|w|^2} = \frac{\pi^2}{M(\Gamma)}, \tag{8.3.2}$$

where $\theta_1(r)$ and $\theta_2(r)$ are the angular coordinates of the ends of the arcs γ_r, and J_g is the Jacobian of g. The replacement of variables is correct in view of Lemma III.3.3 in [152].

Now, we have

$$\pi \le \int\limits_{\gamma_r} |\mathrm{d}\arg g| \le \int\limits_{\gamma_r} |\mathrm{d}\log g| = \int\limits_{\gamma_r} \frac{|\mathrm{d}g(z)|}{|g(z)|} = \int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+re^{\mathrm{i}\theta})|}{|g(z_0+re^{\mathrm{i}\theta})|}\,\mathrm{d}\theta$$

for a.e. $r\in(r_1,r_2)$ and, applying the Schwartz inequality (see, e.g., Theorem I.4 in [29]), we obtain that

$$\begin{aligned}\pi^2 &\le \left(\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+re^{\mathrm{i}\theta})|}{|g(z_0+re^{\mathrm{i}\theta})|}\,\mathrm{d}\theta\right)^2\\ &\le \int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+re^{\mathrm{i}\theta})|^2}{J(z_0+re^{\mathrm{i}\theta})}\,\mathrm{d}\theta \int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{J(z_0+re^{\mathrm{i}\theta})}{|g(z_0+re^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta,\end{aligned}$$

i.e.,

$$\frac{\pi^2}{\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+re^{\mathrm{i}\theta})|^2}{rJ(z_0+re^{\mathrm{i}\theta})}\,\mathrm{d}\theta} \le r\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{J(z_0+re^{\mathrm{i}\theta})}{|g(z_0+re^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta\,.$$

Integrating both sides of the last inequality over $r \in (r_1, r_2)$ gives

$$\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+r\mathrm{e}^{\mathrm{i}\theta})|^2}{rJ(z_0+r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}\theta} \leq \pi^{-2} \int_{r_1}^{r_2} r \int_{\theta_1(r)}^{\theta_2(r)} \frac{J(z_0+r\mathrm{e}^{\mathrm{i}\theta})}{|g(z_0+r\mathrm{e}^{\mathrm{i}\theta})|^2}\mathrm{d}\theta\,\mathrm{d}r. \tag{8.3.3}$$

Combining (8.3.2) and (8.3.3), we have that

$$M(\Gamma) \leq \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{\int\limits_{\theta_1(r)}^{\theta_2(r)} \frac{|g_\theta(z_0+r\mathrm{e}^{\mathrm{i}\theta})|^2}{rJ(z_0+r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}\theta} \right)^{-1} \tag{8.3.4}$$

and by (2.7.7) we obtain (8.3.1). □

Now, combining Lemmas 8.1, 8.2, and 8.4, we obtain following conclusion:

Theorem 8.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a regular homeomorphism of the Sobolev class $W^{1,1}_{\mathrm{loc}}$. Then f is a ring Q-homeomorphism at every point $z_0 \in \partial\mathbb{D}$ with $Q(z) = K^T_\mu(z, z_0)$, $\mu = \mu_f$.*

Corollary 8.1. *Every regular homeomorphism $f : \mathbb{D} \to \mathbb{D}$ of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is a ring Q-homeomorphism at each point $z_0 \in \partial\mathbb{D}$ with $Q(z) = K_\mu(z)$, $\mu = \mu_f$.*

8.4 On Extension of Regular Homeomorphisms to the Boundary

In what follows, for a point $z_0 \in \partial\mathbb{D}$ and a mapping $f : \mathbb{D} \to \mathbb{D}$, we denote

$$C(z_0, f) := \left\{ w \in \mathbb{C} : w = \lim_{n\to\infty} f(z_n), z_n \to z_0 \right\};$$

$C(z_0, f)$ is called a *cluster set* of f at z_0.

Lemma 8.5. *Let $f : \mathbb{D} \to \mathbb{D}$ be a regular homeomorphism such that*

$$\int_{\mathbb{D}(z_0,\varepsilon)} K^T_\mu(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\,\mathrm{d}x\,\mathrm{d}y = o(I^2_{z_0}(\varepsilon)) \tag{8.4.1}$$

as $\varepsilon \to 0$, where $\mu(z) = \mu_f(z)$, $z_0 \in \partial\mathbb{D}$, $D(z_0, \varepsilon) = \{z \in \mathbb{D} : \varepsilon < |z - z_0| < \varepsilon(z_0)\}$, $\varepsilon(z_0) \in (0, 1)$, and $\psi_{z_0,\varepsilon}(t)$ is a family of nonnegative Lebesgue measurable functions on $(0, 1)$ such that

$$0 < I_{z_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,\mathrm{d}t < \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,. \tag{8.4.2}$$

Then f can be extended to the point z_0 by continuity.

Proof. Let us show that the cluster set $E = C(z_0, f)$ is a singleton.

Note that $E \neq \emptyset$ in view of the compactness of $\overline{\mathbb{D}}$ and that $\partial\mathbb{D}$ is strongly accessible at every point $w_0 \in E$. Assume that there is one more point $w^* \in E$. Let $U = B(w_0, r_0) := \{w \in \mathbb{C} : |w - w_0| < r_0\}$, where $0 < r_0 < d(w_0, w^*)$, and $D_m :\; = \{z \in \mathbb{D} : |z - z_0| < 1/m\}$, $m = 1, 2, \ldots$. Then there exist sequences of points w_m and $w_m^* \in F_m = fD_m$ that converge to w_0 and w^*, respectively, for which $|w_0 - w_m| < r_0$ and $|w_0 - w_m^*| > r_0$. The points w_m and w_m^* can be joined by paths C_m in the domains F_m. By the construction,

$$C_m \cap \partial B(w_0, r_0) \neq \emptyset$$

in view of the connectedness of C_m.

Since $\partial\mathbb{D}$ is strongly accessible, there exist a compact set $C \subset \mathbb{D}$ and a number $\delta > 0$ such that

$$M(\Delta(C, C_m, \mathbb{D})) \geq \delta$$

for large m because $\operatorname{dist}(w_0, C_m) \leq |w_0 - w_m| \to 0$ as $m \to \infty$. Note that the set $K = f^{-1}(C)$ is compact as a continuous image of a compact set. Thus, $\varepsilon_0 = \operatorname{dist}(z_0, K) > 0$. Let Γ_ε be the family of all paths joining the disk

$$B_\varepsilon = D(z_0, \varepsilon) := \{z \in \mathbb{D} : |z - z_0| < \varepsilon\},\ \varepsilon \in (0, \varepsilon_0),$$

with the compactum K.

Let $\psi^*_{z_0,\varepsilon}$ be a Borel function such that $\psi^*_{z_0,\varepsilon}(t) = \psi_{z_0,\varepsilon}(t)$ for a.e. $t \in (0, \infty)$, which exists by the Lusin theorem; see, e.g., 2.3.5 in [83]. Then the function

$$\eta_\varepsilon(t) = \begin{cases} \psi^*_{z_0,\varepsilon}(t)/I_{z_0}(\varepsilon), & \text{if } t \in (\varepsilon, \varepsilon_0) \\ 0, & \text{if } t \in \mathbb{R}\setminus(\varepsilon, \varepsilon_0) \end{cases}$$

satisfies the condition

$$\int\limits_{\varepsilon}^{\varepsilon_0} \eta_\varepsilon(t)\,\mathrm{d}t = 1\,.$$

Hence, by Lemmas 8.1, 8.2 and 8.4,

$$M\left(\Gamma_\varepsilon^*\right) \leq \int\limits_{A \cap \mathbb{D}} Q(z) \cdot \eta_\varepsilon^2(|z - z_0|)\,\mathrm{d}m(z),$$

where $\Gamma_\varepsilon^* = \Delta\left(fC_\varepsilon^*, fC_{\varepsilon_0}^*, fA\right)$, $A = A(z_0,\varepsilon,\varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z-z_0| < \varepsilon_0\}$, $C_1^* = S(z_0,\varepsilon)\cap\mathbb{D}$, and $C_2^* = S(z_0,\varepsilon_0)\cap\mathbb{D}$, $Q(z) = K_\mu^T(z,z_0)$. Since $\Gamma_\varepsilon^* < \Gamma_\varepsilon$ and, consequently, $f\Gamma_\varepsilon^* < f\Gamma_\varepsilon$, we have that $M(\Gamma_\varepsilon) \leq M(\Gamma_\varepsilon^*)$, and therefore, $M(\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ by condition (8.4.1).

On the other hand, for every $\varepsilon \in (0,\varepsilon_0)$ and large m, $D_m \subset B_\varepsilon$ and therefore $C_m \subset fB_\varepsilon$. Consequently,

$$M(f\Gamma_\varepsilon) \geq M(\Delta(C,_m;\mathbb{D})) \geq \delta > 0.$$

The obtained contradiction disproves the above assumption that the cluster set E is not a singleton. □

Lemma 8.6. *Let $f:\mathbb{D}\to\mathbb{D}$ be a regular homeomorphism with $K_\mu \in L^1(\mathbb{D})$, where $\mu = \mu_f$. Then*

$$C(z_1,f)\cap C(z_2,f) = \emptyset \qquad \forall\, z_1,z_2 \in \partial\mathbb{D},\ z_1 \neq z_2.$$

Proof. Set $_i = C(z_i,f)$, $i = 1,2$, and $\delta = |z_1 - z_2|$. Let us assume that $E_1\cap E_2 \neq \emptyset$. Set $W_1 = \mathbb{D}\cap D_1$ and $W_2 = \mathbb{D}\setminus\overline{D}_2$, where $D_1 = D(z_1,\frac{\delta}{3})$ and $D_2 = D(z_1,\frac{2\delta}{3})$.

For the function

$$\eta(t) = \begin{cases} \frac{3}{\delta}, & \text{if } t \in \left(\frac{\delta}{3},\frac{2\delta}{3}\right), \\ 0, & \text{if } t \in \mathbb{R}\setminus\left(\frac{\delta}{3},\frac{2\delta}{3}\right), \end{cases}$$

we have that

$$\int\limits_{\frac{\delta}{3}}^{\frac{2\delta}{3}} \eta(t)\,\mathrm{d}t = 1,$$

and by Corollary 8.1, because $K_\mu \in L^1(\mathbb{D})$,

$$M(\Delta(fC_1,fC_2,\mathbb{D})) \leq \int\limits_{A\cap\mathbb{D}} K_\mu(z)\cdot\eta^2(|z-z_0|)\,\mathrm{d}m(z) \leq \frac{9}{\delta^2}\int\limits_{\mathbb{D}} K_\mu(z)\,\mathrm{d}m(z) < \infty$$

for arbitrary continua C_1 and C_2 in $\mathbb{D}$ belonging to the different components of the complement of the ring $A = A\left(z_1,\frac{\delta}{3},\frac{2\delta}{3}\right)$ in $\overline{\mathbb{C}}$.

However, the above estimate contradicts the condition of the weak flatness of the boundary of the unit disk $\mathbb{D}$ if there is $w_0 \in E_1\cap E_2$. Indeed, $w_0 \in \overline{fW_1}\cap\overline{fW_2}$, and there exist paths intersecting any prescribed spheres $\partial B(w_0,r_0)$ and $\partial B(w_0,r_*)$ with small enough radii r_0 and r_* in the domains $W_1^* = fW_1$ and $W_2^* = fW_2$. Hence, the assumption that $E_1\cap E_2 \neq \emptyset$ was not true. □

Corollary 8.2. *For every regular homeomorphism $f:\mathbb{D}\to\mathbb{D}$ with $K_\mu \in L^1(\mathbb{D})$, the inverse homeomorphism f^{-1} admits a continuous extension to $\partial\mathbb{D}$.*

Combining Lemmas 8.5 and 8.6, we obtain the following conclusion:

Lemma 8.7. *Every regular homeomorphism $f:\mathbb{D}\to\mathbb{D}$ is extended to a homeomorphism $\overline{f}:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ provided that $K_\mu\in L^1(\mathbb{D})$ and $K^T_\mu(z,z_0)$ satisfies the condition (8.4.1) at every point $z_0\in\partial\mathbb{D}$, where $\mu(z)=\mu_f(z)$.*

8.5 Existence Theorems for the Dirichlet Problem

Every analytic function f in a domain D satisfies to the simplest Beltrami equation

$$f_{\bar z}=0 \tag{8.5.1}$$

when $\mu(z)\equiv 0$. If an analytic function f defined in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ is continuous in its closure, then by the Schwarz formula

$$f(z)=\mathrm{i}\,\mathrm{Im}\,f(0)+\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\mathrm{Re}\,f(\zeta)\cdot\frac{\zeta+z}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta}\,; \tag{8.5.2}$$

see, e.g., Sect. 8, Chap. III, Part 3 in [111], p. 346. Thus, the analytical function f in the unit disk $\mathbb{D}$ is defined, up to a purely imaginary number ic, $c=\mathrm{Im}\,f(0)$, by its real part $\varphi(\zeta)=\mathrm{Re}\,f(\zeta)$ on the boundary of the unit disk.

The *Dirichlet problem* for the Beltrami equation (B) in a domain $D\subset\mathbb{C}$ is the problem of the existence of a continuous function $f:D\to\mathbb{C}$ having partial derivatives of the first order a.e., satisfying (B) a.e. and such that

$$\lim_{z\to\zeta}\mathrm{Re}\,f(z)=\varphi(\zeta)\qquad\forall\zeta\in\partial D \tag{8.5.3}$$

for a prescribed continuous function $\varphi:\partial D\to\mathbb{R}$. It is obvious that if f is a solution of such a problem, then the function $F(z)=f(z)+\mathrm{i}c$, $c\in\mathbb{R}$, is also.

The *regular solution* of the Dirichlet problem (8.5.3) with $\varphi(\zeta)\equiv c$, $\zeta\in\partial D$, for the Beltrami equation (B) is the function $f(z)\equiv c$, $z\in D$. If $\varphi(\zeta)\not\equiv\mathrm{const}$, then the regular solution of such a problem is a continuous, discrete, and open mapping $f:D\to\mathbb{C}$ of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ with its Jacobian $J_f(z)=|f_z|^2-|f_{\bar z}|^2\neq 0$ a.e. satisfying the condition (8.5.3) and (B) a.e. If $D=\mathbb{D}$, we in addition assume that $\mathrm{Im}\,f(0)=0$.

Recall that a mapping $f:D\to\mathbb{C}$ is called *discrete* if the preimage of $f^{-1}(y)$ consists of isolated points for every $y\in\mathbb{C}$ and *open* if f maps every open set $U\subseteq D$ onto an open set in $\mathbb{C}$.

In what follows, we prove that a regular solution of the Dirichlet problem (8.5.3) exists for every continuous function $\varphi:\partial\mathbb{D}\to\mathbb{R}$ for wide classes of the degenerate Beltrami equations (B) in the unit disc $\mathbb{D}$. The Dirichlet problem in the unit disk is a model. The case of an arbitrary bounded simply connected domain D in $\mathbb{C}$ can be reduced to the case of the unit disk by using conformal Riemann mapping theorem.

Lemma 8.8. *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function. If $K_\mu \in L^1(\mathbb{D})$ and $K_\mu^T(z,z_0)$ satisfies the condition (8.4.1) at every point $z_0 \in \overline{\mathbb{D}}$, then the Beltrami equation (B) has a regular solution f of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.*

Proof. By Stoilow's factorization theorem (see, e.g., [241]), every continuous open discrete mapping $f : \mathbb{D} \to \mathbb{C}$ can be presented as

$$f = h \circ g, \tag{8.5.4}$$

where g is a homeomorphism of $\mathbb{D}$ onto $\mathbb{D}$ and h is an analytical function in $\mathbb{D}$. Hence, we find the regular solution of the Dirichlet problem (8.5.3) in the form (8.5.4).

Under the condition (8.4.1), by Lemma 7.1 and the Riemann theorem, there is a regular homeomorphism $g : \mathbb{D} \to \mathbb{D}$, $g(0) = 0$, which is a $W^{1,1}_{\mathrm{loc}}$ solution for the Beltrami equation (B), and g by Lemma 8.7 can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$.

By the Schwarz and Poisson formulas (see, e.g., Sects. 8 and 10, Chap. III, Part 3 in [111]), the analytic function h in (8.5.4) with $\mathrm{Im}\, h(0) = 0$ can be calculated in $\mathbb{D}$ through its real part on the boundary:

$$h(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=1} \mathrm{Re}\, f \circ g^{-1}(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{\mathrm{d}\zeta}{\zeta}\,. \tag{8.5.5}$$

□

By Lemmas 5.1 and 8.8 with $\psi_{z_0,\varepsilon}(t) \equiv 1/t \log \frac{1}{t}$, we obtain the following:

Theorem 8.3. *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function such that K_μ has the* BMO *majorant. Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.*

Remark 8.2. In particular, we have that if

$$K_\mu(z) \le Q(z) \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}), \tag{8.5.6}$$

then the conclusion of the last theorem holds because $W^{1,2}(\mathbb{D}) \subset VMO(\mathbb{D})$; see, e.g., [58]. Moreover, in this case $f \in W^{1,2}_{\mathrm{loc}}$ (see, e.g., [158]), and the problem has the unique regular solution for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

By Lemmas 5.3 and 8.8, this theorem admits the essential generalization.

Theorem 8.4. *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function such that $K_\mu \in L^1(\mathbb{D})$ and $K_\mu^T(z,z_0) \le Q(z,z_0)$, where the function $Q(z,z_0)$ belongs to* FMO(z_0) *in z for all $z_0 \in \overline{\mathbb{D}}$. Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.*

Corollary 8.3. *Let* $\mu : \mathbb{D} \to \mathbb{D}$ *be a measurable function such that*

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \leq Q(z) \in \mathrm{FMO}\left(\overline{\mathbb{D}}\right). \tag{8.5.7}$$

Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

Corollary 8.4. *Let* $\mu : \mathbb{D} \to \mathbb{D}$ *be a measurable function with* $K_\mu \in L^1(\mathbb{D})$ *such that*

$$\overline{\lim_{\varepsilon\to 0}} \;\; -\!\!\!\!\!\!\int_{\mathbb{D}(z_0,\varepsilon)} \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\,\mu(z)\right|^2}{1-|\mu(z)|^2}\; \mathrm{d}m(z) < \infty \qquad \forall\, z_0 \in \overline{\mathbb{D}}, \tag{8.5.8}$$

where $\mathbb{D}(z_0,\varepsilon) = \mathbb{D} \cap B(z_0,\varepsilon)$, $B(z_0,\varepsilon) = \{z \in \mathbb{C} : |z-z_0| < \varepsilon\}$. *Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function* $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

Corollary 8.5. *Let* $\mu : \mathbb{D} \to \mathbb{D}$ *be a measurable function such that*

$$k_{z_0}(\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \forall z_0 \in \overline{\mathbb{D}} \tag{8.5.9}$$

as $\varepsilon \to 0$, *where* $k_{z_0}(\varepsilon)$ *is the average of the function* $K^T_\mu(z,z_0)$ *over* $\mathbb{D}\bigcap S(z_0,\varepsilon)$, $S(z_0,\varepsilon) = \{z \in \mathbb{C} : |z-z_0| = \varepsilon\}$. *Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function* $\varphi : \mathbb{D} \to \mathbb{R}$.

Remark 8.3. In particular, the conclusion holds if

$$K^T_\mu(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\, z_0 \in \overline{\mathbb{D}}. \tag{8.5.10}$$

Theorem 8.5. *Let* $\mu : \mathbb{D} \to \mathbb{D}$ *be a measurable function with* $K_\mu \in L^1(\mathbb{D})$ *such that*

$$\int\limits_0^{\delta(z_0)} \frac{\mathrm{d}r}{\|K^T_\mu\|_1(r)} = \infty \qquad \forall\, z_0 \in \overline{\mathbb{D}}, \tag{8.5.11}$$

where $\|K^T_\mu\|_1(r) = \int_{\gamma_r} K^T_\mu(z,z_0)|\mathrm{d}z|$ *is the* L^1 *norm of the function* $K^T_\mu(z,z_0)$ *over the arcs* $\gamma_r = \mathbb{D} \cap S(z_0,r)$ *of the circles* $S(z_0,r) = \{z \in \mathbb{C} : |z-z_0| = r\}$, $0 < \delta(z_0) < 1$. *Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function* $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

Theorem 8.5 follows from Lemma 8.8 under the special choice

$$\psi_{z_0,\varepsilon}(t) \equiv \psi_z(t) = \begin{cases} 1/\left[tk_{z_0}(t)\right], & t \in (0,\varepsilon_0), \\ 0, & t \in [\varepsilon_0,\infty), \end{cases}$$

where $\varepsilon_0 = \delta(z_0)$ and $k_{z_0}(t)$ is the mean value of $K_\mu^T(z,z_0)$ over $\gamma_t = \mathbb{D} \cap S(z_0,t)$.

Remark 8.4. Of course, all the above results on the existence of regular solutions of the Dirichlet problem for the Beltrami equation (B) can be formulated in terms of the dilatation $K_\mu(z)$ because $K_\mu^T(z,z_0) \le K_\mu(z)$. In particular, the latter theorem big $K_\mu \in L^p(\mathbb{D})$, $p > 1$, instead of $K_\mu^T(z,z_0)$ can be derived from Theorem 2 in [174] big applying Lemma 8.7 too. Note also that the boundary behavior of mappings with bounded Dirichlet integral was studied in Suvorov's works; see, e.g., [248, 249].

8.6 Examples of the Dirichlet Problem

Example 1. Let

$$\mu(z) = k(|z|)\,\frac{z}{\bar{z}}, \tag{8.6.1}$$

where $k(\tau) : \mathbb{R} \to (-1,1)$ is a measurable function such that

$$\int_\varepsilon^1 \frac{1+k(\tau)}{1-k(\tau)}\,\frac{\mathrm{d}\tau}{\tau} < \infty \qquad \forall\, \varepsilon \in (0,1) \tag{8.6.2}$$

and

$$\int_0^1 \frac{1+k(\tau)}{1-k(\tau)}\,\frac{\mathrm{d}\tau}{\tau} = \infty. \tag{8.6.3}$$

Then a regular solution of the Dirichlet problem (8.5.3) for the Beltrami equation (B) can be written in the explicit form

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \operatorname{Re}\varphi(\zeta) \cdot \frac{\zeta + \omega(z)}{\zeta - \omega(z)} \cdot \frac{\mathrm{d}\zeta}{\zeta}, \tag{8.6.4}$$

where

$$\omega(z) = \frac{z}{|z|} \exp\left(- \int_{|z|}^1 \frac{1+k(\tau)}{1-k(\tau)}\,\frac{\mathrm{d}\tau}{\tau} \right) \tag{8.6.5}$$

is a regular homeomorphic solution of Beltrami equation (B) with μ from (8.6.1); see Propositions 6.4 and 6.5 in [213].

Example 2. The example of a complex dilatation μ which is given further in (8.6.7) shows that the Dirichlet problem (8.5.3) for the Beltrami equation (B) has no regular solution for every $\varphi \neq$ const, even in the case of $\mu \in C^\infty(\overline{\mathbb{D}} \setminus \{0\})$, $K_\mu \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{D}} \setminus \{0\})$ and $K_\mu \in L^p(\mathbb{D})$ for any arbitrarily large $p \geq 1$. Moreover, K_μ has only one singularity at zero and none on the boundary $\partial\mathbb{D}$.

In the monograph [165], the following simple example of the mapping was given:

$$g(z) = \frac{z}{|z|}\,(1+|z|^\alpha) = \mathrm{e}^{\mathrm{i}\vartheta}\,(1+r^\alpha)\,, \tag{8.6.6}$$

$z = x+\mathrm{i}y = r\mathrm{e}^{\mathrm{i}\vartheta}, \alpha \in (0,1)$, that maps the punctured unit disk $\mathbb{D}_0 = \mathbb{D}\setminus\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the ring $A(0,1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$, and thus, g has the unremovable singularity at zero.

Let us calculate the complex dilatation of the mapping g. By (7.5.10) and (7.5.11) note that

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial z}\cdot\frac{\partial z}{\partial r} + \frac{\partial g}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial r} = \frac{z}{|z|}\cdot g_z + \frac{\overline{z}}{|z|}\cdot g_{\overline{z}}\,,$$

and

$$\frac{\partial g}{\partial \vartheta} = \frac{\partial g}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} + \frac{\partial g}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \vartheta} = \mathrm{i}\,(zg_z - \overline{z}g_{\overline{z}})\,.$$

Thus,

$$zg_z = \frac{1}{2}\left(r\frac{\partial g}{\partial r} - \mathrm{i}\frac{\partial g}{\partial \vartheta}\right) = \frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2}\,(1+(1+\alpha)r^\alpha)\,,$$

and

$$\overline{z}g_{\overline{z}} = \frac{1}{2}\left(r\frac{\partial g}{\partial r} + \mathrm{i}\frac{\partial g}{\partial \vartheta}\right) = -\frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2}\,(1+(1-\alpha)r^\alpha)\,,$$

and consequently

$$\mu(z) = \frac{f_{\overline{z}}}{f_z} = -\mathrm{e}^{2\mathrm{i}\vartheta}\cdot\frac{1+(1-\alpha)r^\alpha}{1+(1+\alpha)r^\alpha}. \tag{8.6.7}$$

Then under $\alpha \in (0,1)$, the dilatation

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} = \frac{1+r^\alpha}{\alpha r^\alpha} = \frac{1}{\alpha}\left(1+r^{-\alpha}\right)$$

has only one singularity as $r \to 0$ and, moreover, $K_\mu \in L^p(\mathbb{D})$ for every $p < 2/\alpha$. Thus, under $\alpha \to 0$, the degree of integrability of K_μ can be arbitrarily large. Moreover, $K_\mu \in L^\infty_{\mathrm{loc}}(\mathbb{D}_0)$ and $\mu \in C^\infty\left(\overline{\mathbb{D}}\setminus\{0\}\right)$, i.e., f is locally quasiconformal in the punctured disk $\mathbb{D}_0$.

It is easy to show that under μ from (8.6.7), the Dirichlet problem (8.5.3) for (B) has no regular solutions f with any continuous function $\varphi \neq \text{const}$. Indeed, by Stoilow's factorization theorem $f = A \circ h$, where A is an analytic function and h is a homeomorphism. However, h is a locally quasiconformal mapping outside of some discrete set of points M in $\mathbb{D}_0$, because branch points of the analytic function A are isolated, and satisfies the Beltrami equation (B) a.e. Thus, by the uniqueness theorem for quasiconformal mappings we have that $h|_{\mathbb{D}_0\setminus M} = C \circ g|_{\mathbb{D}_0\setminus M}$, where C is a conformal mapping in $g(\mathbb{D}_0 \setminus M) = g(\mathbb{D}_0) \setminus g(M)$. Note that, since g is a homeomorphism in $\mathbb{D}_0$, the set $g(M)$ consists of isolated points and thus is removable for the conformal mapping C. However, then M is removable for h and $h = C \circ g$. Consequently, $f = A_* \circ g$ where A_* is analytic function.

By the definition, the regular solution f is continuous at zero. However, $g^{-1}(w) \to 0$ as $|w| \to 1$. Thus, $A_*(w) \to f(0) = \mathrm{Re} f(0)$ as $|w| \to 1$ and by the Riemann–Schwarz symmetry principle (see, e.g., Theorem 3, Part 3, Chap. 4, Sect. 5 in [111]); A_* is extended through the unit circle $|w| = 1$. Simultaneously, $A_*(w) \equiv f(0)$ on this circle and then $A_* \equiv \text{const}$ (see, e.g., Theorem 2, Part 3, Chap. 3, Sect. 2 in [111]), which is not possible for $\varphi \neq \text{const}$.

Example 3. The example of the mapping (8.6.6) allows us to construct another example of a complex dilatation μ below in (8.6.8) of the class $C^\infty(\mathbb{D}) \setminus \{0\}$ with $K_\mu \in L^\infty_{\text{loc}}(\mathbb{D})$, for which the Beltrami equation (B) has no regular solutions of the Dirichlet problem (8.5.3) under $\varphi \neq \text{const}$; K_μ has singularities only on $\partial\mathbb{D}$.

Note that the relation (8.6.6) allows us to define the homeomorphism $w = g(z)$ mapping $\mathbb{C}_* = \overline{\mathbb{C}} \setminus \{0\}$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $g(\infty) = \infty$; in polar coordinates $R = |w| = 1 + r^\alpha$, $\theta = \arg w = \arg z = \vartheta$. Consider the mapping

$$G = j \circ g^{-1} \circ j,$$

where j is inversion with respect to the unit circle in $\mathbb{C}$. By the construction, $\kappa = G(\varsigma)$ maps $\mathbb{D}$ onto $\mathbb{C}$ and $G(\varsigma) \to \infty$ as $|\varsigma| \to 1$. In view of the circle symmetry of the mapping g,

$$G = j_* \circ g^{-1} \circ j_*,$$

where $j_*(\varsigma) = 1/\varsigma$ is a conformal mapping.

It is known that

$$\mu_{g^{-1}} = -\nu_g \circ g^{-1}, \quad \nu_{g^{-1}} = -\mu_g \circ g^{-1},$$

where

$$\nu_g(z) := \frac{g_{\overline{z}}}{\overline{g_z}} = -\mathrm{e}^{2i\vartheta} \cdot \frac{1 + (1-\alpha)r^\alpha}{1 + (1+\alpha)r^\alpha} = \mu_g(z);$$

see, e.g., IC(4) in [9]. In view of the latter, $\mu_{g^{-1}} = \nu_{g^{-1}}$ and then, by IC(7) and (9) in [9],

$$\mu_G = -\mu_g \circ g^{-1} \circ j_*,$$

i.e.,

$$\mu_G(\varsigma) = \mathrm{e}^{-2\mathrm{i}\gamma} \cdot \frac{1+(1-\alpha)[\frac{1}{\rho}-1]}{1+(1+\alpha)[\frac{1}{\rho}-1]} = \mathrm{e}^{-2\mathrm{i}\gamma} \cdot \frac{\rho+(1-\alpha)(1-\rho)}{1+(1+\alpha)(1-\rho)}, \tag{8.6.8}$$

where $\rho = |\varsigma|$ and $\gamma = \arg\varsigma$. Consequently, we have that

$$K_{\mu_G}(\varsigma) = \frac{1+\left(\frac{1}{\rho}-1\right)}{\alpha\left(\frac{1}{\rho}-1\right)} = \frac{1}{\alpha(1-\rho)} = \frac{1}{\alpha(1-|\varsigma|)}\,.$$

Let us show that the Beltrami equation (B) has no regular solutions f of the Dirichlet problem (8.5.3) for $\varphi \neq \text{const}$ and $\mu = \mu_G$. Let us assume that there is a regular solution f. Then, arguing as in Example 2, we obtain that $f = A \circ G$ in $\mathbb{D}$, where A is an analytic function in $G(\mathbb{D}) = \mathbb{C}$. However, the cluster set

$$C(\partial\mathbb{D}, \operatorname{Re} f) = \{w \in \overline{\mathbb{C}} : w = \lim_{n\to\infty} \operatorname{Re} f(z_n),\ |z_n| \to 1\} = \varphi(\partial\mathbb{D})$$

is bounded, and $G(\varsigma) \to \infty$ if and only if $|\varsigma| \to 1$. Thus, $\operatorname{Re}A$ is bounded and, consequently, $\operatorname{Re}A = \text{const}$ by the Liouville theorem for harmonic functions (see, e.g., Theorem 5, Chap. 5, Sect. 14 in [229]), which is not possible for $\varphi \neq \text{const}$.

8.7 On Integral Conditions in the Dirichlet Problem

On the basis of Corollary 2.7, we derive the following consequence of Theorem 8.5:

Theorem 8.6. *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(\mathbb{D})$ and such that*

$$\int\limits_{D(z_0)} \Phi_{z_0}\left(K^T_\mu(z, z_0)\right)\, \mathrm{d}m(z) \ <\ \infty \qquad \forall\, z_0 \in \overline{\mathbb{D}}, \tag{8.7.1}$$

where $D(z_0) = \{z \in \mathbb{D} : |z - z_0| < \delta(z_0)\}$, $0 < \delta(z_0) < 1$ and $\Phi_{z_0} : [0,\infty] \to [0,\infty]$ is a nondecreasing convex function that satisfies at least one of the conditions (2.8.12)–(2.8.17).

Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

Next, since $K^T_\mu(z, z_0) \leq K_\mu(z)$, Theorem 8.6 implies the following simple but very important consequence:

Corollary 8.6. *Let $\mu : \mathbb{D} \to \mathbb{D}$ be a measurable function such that*

$$\int_{\mathbb{D}} \Phi(K_\mu(z))\, \mathrm{d}m(z) \;<\; \infty, \tag{8.7.2}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a nondecreasing convex function that satisfies at least one of the conditions (2.8.12)–(2.8.17).

Then the Beltrami equation (B) has a regular solution of the Dirichlet problem (8.5.3) for every continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

Remark 8.5. Note that conditions (2.8.12)–(2.8.17) are not only sufficient but also necessary to have a regular solution of the Dirichlet problem (8.5.3) (under every continuous nonconstant function $\varphi : \partial\mathbb{D} \to \mathbb{R}$) for all Beltrami equations (B) with integral constraints (8.7.2). Indeed, by the Stoilow factorization (see, e.g., [241]), such a regular solution f should have the representation $f = h \circ g$, where h is a conformal mapping and g is a homeomorphic solution of the class $W^{1,1}_{\mathrm{loc}}(\mathbb{D})$ to the Beltrami equation (B). However, these conditions are necessary for the existence of such solutions of all Beltrami equations (B) with integral constraints (8.7.2) as it was shown in Sect. 8.5.

Chapter 9
On the Beltrami Equations with Two Characteristics

9.1 Introduction

The Beltrami equations of the first type

$$f_{\bar{z}} = \mu(z) f_z \tag{9.1.1}$$

is the basic equation for the theory of quasiconformal mapping in the complex plane; see, e.g., [9, 30, 44] and [152]. The well-known measurable mapping theorem solves the problem on the existence and uniqueness for the classical case.

The existence problem for the Beltrami equations with two characteristics,

$$f_{\bar{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z}, \tag{9.1.2}$$

where $|\mu(z)| + |\nu(z)| < 1$ a.e., was also resolved first in the case of the bounded dilatations

$$K_{\mu,\nu}(z) \ := \ \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \ ; \tag{9.1.3}$$

see Theorem 5.1 in [44]. Recently, in [48], the existence of homeomorphic solutions of the class $W^{1,s}_{\text{loc}}$, $s \in [1,2)$, for (9.1.2) was stated in the case when $K_{\mu,\nu}$ had a majorant Q in the class BMO, bounded mean oscillation, by John and Nirenberg; see, e.g., [122, 200]. Note that $L^{\infty} \subset \text{BMO} \subset L^p_{\text{loc}}$ for all $p \in [1,\infty)$. This section is devoted to the study of more general cases; see [49].

If $f : D \to \mathbb{C}$ is a homeomorphic ACL solution of the Beltrami equation (9.1.2) with $K_{\mu,\nu} \in W^{1,1}_{\text{loc}}(D)$, then $f \in W^{1,1}_{\text{loc}}(D)$. Furthermore, if $K_{\mu,\nu} \in W^{1,p}_{\text{loc}}(D)$, $p \in [1,\infty]$, then $f \in W^{1,s}_{\text{loc}}(D)$ where $s = 2p/(p+1)$; see Proposition 2.4. The condition $f^{-1} \in W^{1,2}_{\text{loc}}(f(D))$ in the proof of our existence theorems for (9.1.2) further implies that a.e. point z is a *regular point* for the mapping f, i.e., f is differentiable at z and $J_f(z) \neq 0$; see Proposition 2.5. Conversely, if $f \in W^{1,1}_{\text{loc}}(D)$, $K_{\mu,\nu} \in L^1_{\text{loc}}(D)$ and $J_f(z) \neq 0$ a.e., then $f^{-1} \in W^{1,2}_{\text{loc}}(f(D))$; see, e.g., [105].

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We call a homeomorphism $f \in W^{1,1}_{\rm loc}(D)$ a *regular solution* of (9.1.2) if f satisfies (9.1.2) a.e. and $J_f(z) \neq 0$ a.e. If, in addition, f admits a homeomorphic extension to $\overline{\mathbb{C}}$ with $f(\infty) = \infty$ and $f \in W^{1,1}_{\rm loc}(\mathbb{C})$ which is conformal in $\overline{\mathbb{C}} \setminus \overline{D}$, then we call f a *superregular solution* of (9.1.2).

In the classical case, when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty(D)$, every ACL homeomorphic solution f of the Beltrami equation (9.1.1) is in the class $W^{1,2}_{\rm loc}(D)$ with $f^{-1} \in W^{1,2}_{\rm loc}(f(D))$. In the case $\|\mu\|_\infty = 1$ with $K_\mu \leq Q \in {\rm BMO}$, again $f^{-1} \in W^{1,2}_{\rm loc}(f(D))$ and f belongs to $W^{1,s}_{\rm loc}(D)$ for all $1 \leq s < 2$, but already not necessarily to $W^{1,2}_{\rm loc}(D)$; see [212]. However, there is a number of degenerate Beltrami equations (9.1.2) for which there exist homeomorphic solutions f of the class $W^{1,1}_{\rm loc}(D)$ with $f^{-1} \in W^{1,2}_{\rm loc}(f(D))$, as shown below.

Later on, we use the notations $B(z_0, r) : = \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$, $\mathbb{B}(r) : = B(0, r)$ and $\mathbb{B} : = \mathbb{B}(1)$.

9.2 One Convergence Theorem

Our approach in deriving criteria for existence of regular solutions to (9.1.2) is approximative and based on the corresponding convergence theorems below and on the Arzela–Ascoli theorem combined with moduli techniques under estimates of the distance distortion.

The following theorem is a generalization of the convergence result in [44], Lemma 4.2, where Q was in L^∞:

Theorem 9.1. *Let D be a domain in $\mathbb{C}$ and let $f_n : D \to \mathbb{C}$ be a sequence of homeomorphic $W^{1,1}_{\rm loc}(D)$ solutions of the equations $\overline{\partial} f_n = \mu_n \partial f_n + \nu_n \overline{\partial f_n}$ with $|\mu_n(z)| + |\nu_n(z)| < 1$ a.e. such that*

$$\frac{1 + |\mu_n(z)| + |\nu_n(z)|}{1 - |\mu_n(z)| - |\nu_n(z)|} \leq Q(z) \in L^1_{\rm loc}(D) \qquad \forall\, n = 1, 2, \ldots . \tag{9.2.1}$$

If $f_n \to f$ uniformly on each compact set in D where $f : D \to \mathbb{C}$ is a homeomorphism, then $f \in W^{1,1}_{\rm loc}(D)$ and $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^1_{\rm loc}(D)$. Moreover, if in addition $\mu_n \to \mu$ and $\nu_n \to \nu$ a.e., then $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$ a.e.

Proof. By Lemma 2.11, to prove the first part of the theorem, it suffices to show that ∂f_n and $\overline{\partial} f_n$ are uniformly bounded in $L^1_{\rm loc}(D)$ and have locally absolute equicontinuous indefinite integrals. So, let C be a compact set in D, and let V be an open set with their compact closure $\overline{V}$ in D such that $C \subset V$, say $V = \{z \in D : {\rm dist}(z, C) < r\}$ where $r < {\rm dist}(C, \partial D)$. Note that

$$|\,\overline{\partial} f_n| \leq |\,\partial f_n| \leq |\,\partial f_n| + |\,\overline{\partial} f_n| \leq Q^{1/2}(z) \cdot J_n^{1/2}(z) \qquad \text{a.e.,}$$

where J_n is the Jacobian of f_n. Consequently, by the Hölder inequality and Lemma III.3.3 in [152]

$$\int_E |\,\partial f_n|\;\mathrm{d}m(z) \le \left|\int_E Q(z)\;\mathrm{d}m(z)\right|^{1/2} \;|\,f_n(C)\,|^{1/2}$$

for every measurable set $E \subseteq C$. Hence, by the uniform convergence of f_n to f on C,

$$\int_E |\,\partial f_n|\;\mathrm{d}m(z) \;\le\; \left|\int_E Q(z)\;\mathrm{d}m(z)\right|^{1/2} \;|\,f(V)\,|^{1/2} \tag{9.2.2}$$

for large enough n, and thus, the first part of the proof is complete.

Now, assume that $\mu_n(z) \to \mu(z)$ and $\nu_n(z) \to \nu(z)$ a.e. Set $\zeta(z) = \overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\overline{\partial f(z)}$ and show that $\zeta(z) = 0$ a.e. Indeed, for every disk B with $\overline{B} \subset D$, by the triangle inequality

$$\left|\int_B \zeta(z)\,\mathrm{d}x\,\mathrm{d}y\right| \;\le\; I_1(n) + I_2(n) + I_3(n),$$

where

$$I_1(n) = \left|\int_B \left(\overline{\partial} f(z) - \overline{\partial} f_n(z)\right)\,\mathrm{d}x\,\mathrm{d}y\right|,$$

$$I_2(n) = \left|\int_B (\mu(z)\,\partial f(z) - \mu_n(z)\,\partial f_n(z))\,\mathrm{d}x\,\mathrm{d}y\right|$$

and

$$I_3(n) = \left|\int_B \left(\nu(z)\,\overline{\partial f(z)} - \nu_n(z)\,\overline{\partial f_n(z)}\right)\,\mathrm{d}x\,\mathrm{d}y\right|.$$

Note that $I_1(n) \to 0$ because $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^1_{\mathrm{loc}}(D)$ by the first part of the proof. Next, $I_3(n) \le I_3'(n) + I_3''(n)$, where

$$I_3'(n) = \left|\int_B \overline{\nu(z)}(\partial f(z) - \partial f_n(z))\,\mathrm{d}x\,\mathrm{d}y\right|$$

and

$$I_3''(n) = \left|\int_B \overline{(\nu(z) - \nu_n(z))}\partial f_n(z)\,\mathrm{d}x\,\mathrm{d}y\right|.$$

Again, by the weak convergence $\partial f_n \to \partial f$ in $L^1_{\rm loc}(D)$, we have that $I_3'(n) \to 0$ because $\nu \in L^\infty(D)$. Moreover, given $\varepsilon > 0$, by (9.2.2)

$$\int_E |\,\partial f_n(z)|\,\mathrm{d}m(z) < \varepsilon, \quad n = 1, 2, \ldots, \tag{9.2.3}$$

whenever E is every measurable set in B with $|E| < \delta$ for small enough $\delta > 0$.

Further, by the Egoroff theorem (see, e.g., III.6.12 in [73]), $\nu_n(z) \to \nu(z)$ uniformly on some set $S \subset B$ such that $|E| < \delta$, where $E = B \setminus S$. Hence $|\nu_n(z) - \nu(z)| < \varepsilon$ on S and, by (9.2.3),

$$\begin{aligned} I_3''(n) &\le \varepsilon \int_S |\,\partial f_n(z)|\,\mathrm{d}m(z) + 2\int_E |\,\partial f_n(z)|\,\mathrm{d}m(z) \\ &\le \varepsilon \left\{ 2 + \left(\int_B Q(z)\,\mathrm{d}m(z) \right)^{1/2} |\,f(\lambda B)|^{1/2} \right\} \end{aligned}$$

for some $\lambda > 1$ and for all large enough n, i.e., $I_3''(n) \to 0$ because $\varepsilon > 0$ is arbitrary. The fact that $I_2(n) \to 0$ is proved similarly. Thus, $\int_B \zeta(z)\,\mathrm{d}x\,\mathrm{d}y = 0$ for all disks B with $\overline{B} \subset D$. Finally, by the Lebesgue theorem on differentiability of indefinite integral (see, e.g., IV(6.3) in [224]), $\zeta(z) = 0$ a.e. in D. □

Corollary 9.1. *Let D be a domain in $\mathbb{C}$ and $f_n : D \to \mathbb{C}$, $n = 1, 2, \ldots$, be a sequence of quasiconformal mappings which satisfy the equations $\overline{\partial} f_n = \mu_n \partial f_n + \nu_n \overline{\partial f_n}$ with $|\mu_n(z)| + |\nu_n(z)| < 1$ a.e. and the condition (9.2.1). If $f_n \to f$ locally uniformly with respect to the spherical (chordal) metric, then either f is constant or f is a homeomorphism of the class $W^{1,1}_{\rm loc}(D)$ and $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^1_{\rm loc}(D)$. Moreover, if in addition $\mu_n \to \mu$ and $\nu_n \to \nu$ a.e., then $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$ a.e.*

Proof. Note that if f satisfies (9.1.2), then f also satisfies (9.1.1) with

$$\mu_*(z) = \mu(z) + \nu(z) \frac{\overline{f_z}}{f_z}$$

setting here $\mu_*(z) = \mu(z)$ at points $z \in D$ where $f_z = 0$. Hence, we may apply to (9.1.2) the first part of Corollary 2.3 in Sect. 2.6. Thus, combining it with Proposition 2.7 and Remark 2.8, we obtain the statement from Theorem 9.1. □

9.3 The Main Lemma

The following lemma is the main tool for obtaining many criteria of existence of regular solutions for the Beltrami equations with two characteristics:

Lemma 9.1. *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu(z)| < 1$ a.e. and $K_{\mu,\nu} \in L^1_{\rm loc}(D)$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 < dist(z_0, \partial D)$ and a one-parameter family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, such that*

$$0 \;<\; I_{z_0}(\varepsilon) \;:\; = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,{\rm d}t \;<\; \infty\,, \tag{9.3.1}$$

and such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,{\rm d}m(z) \;=\; o(I^2_{z_0}(\varepsilon)) \tag{9.3.2}$$

as $\varepsilon \to 0$. Then the Beltrami equation (9.1.2) has a regular solution.

Proof. Setting $D_n = D \cap \mathbb{B}(n)$ and

$$\mu_n(z) \;=\; \begin{cases} \mu(z)\,, & \text{if } z \in D_n,\ K_{\mu,\nu}(z) \le n \\ 0\,, & \text{otherwise in } \mathbb{B}(n) \end{cases} \tag{9.3.3}$$

and

$$\nu_n(z) \;=\; \begin{cases} \nu(z)\,, & \text{if } z \in D_n,\ K_{\mu,\nu}(z) \le n \\ 0\,, & \text{otherwise in } \mathbb{B}(n) \end{cases} \tag{9.3.4}$$

we have that $K_{\mu_n,\nu_n}(z) \le n$ in $\mathbb{B}(n)$. Then by Theorem 5.1 in [44], there exist n-quasiconformal mappings $\omega_n : \mathbb{B} \to \mathbb{B}$ with $\omega_n(0) = 0$ that satisfy the equations $\overline{\partial}\omega_n = \mu_n^*\partial\omega_n + \nu_n^*\overline{\partial\omega_n}$ a.e., where $\mu_n^*(z) = \mu_n(nz)$ and $\nu_n^*(z) = \nu_n(nz)$, $z \in \mathbb{B}$. Note that $f_n(z) = \omega_n(z/n)/|\omega_n(1/n)|$ map $\mathbb{B}(n)$ onto $\mathbb{B}(R_n)$ for $R_n = 1/|\omega_n(1/n)| \ge 1$, $f_n(0) = 0$, $|f_n(1)| = 1$ and f_n satisfy the equations $\overline{\partial} f_n = \mu_n\partial f_n + \nu_n\overline{\partial f_n}$ a.e. in $\mathbb{B}(n)$. We extend f_n over $\partial\mathbb{B}(n)$ and $\partial\mathbb{B}(R_n)$ to $\overline{\mathbb{C}}$ by the symmetry principle. The latter implies, in particular, that $f_n(\infty) = \infty$.

Let Γ_ε be a family of all paths joining the circles $C_\varepsilon = \{z \in \mathbb{C} : |z-z_0| = \varepsilon\}$ and $C_0 = \{z \in \mathbb{C} : |z-z_0| = \varepsilon_0\}$ in the ring $A_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z-z_0| < \varepsilon_0\}$. Let also ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in (0,\infty)$. Such a function ψ^* exists by the theorem of Lusin; see, e.g., [224], p. 69. Then the function

$$\rho_\varepsilon(z) = \begin{cases} \psi^*(|z-z_0|)/I_{z_0}(\varepsilon), & \text{if } z \in A_\varepsilon \\ 0, & \text{if } z \in \mathbb{C}\backslash A_\varepsilon \end{cases}$$

is admissible for Γ_ε. Hence, by (2.5.6) applied to the restrictions $h_n = f_n|_{A_\varepsilon}$,

$$M(f_n\Gamma_\varepsilon) \le \int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z)\cdot{\rho_\varepsilon}^2(|z-z_0|)\,{\rm d}x{\rm d}y\,,$$

and, in view of (9.3.2), $M(f_n\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to the parameter $n = 1,2,\ldots$. Thus, in view of the normalization $f_n(0) = 0$, $|f_n(1)| = 1$

and $f_n(\infty)=\infty$, the sequence f_n is equicontinuous in D by Lemma 2.13 with $\delta(F)=1/\sqrt{2}$. Consequently, by the Arzela–Ascoli theorem (see, e.g., [73], p. 382, and [74], p. 267), it has a subsequence, denoted again by f_n, which converges locally uniformly with respect to the spherical (chordal) metric to some nonconstant mapping f in D. Then by Corollary 9.1, f is a homeomorphic solution of the class $W^{1,1}_{\text{loc}}(D)$ to (9.1.2).

By Corollary 2.4 from Sect. 2.4 it follows that $f^{-1}\in W^{1,2}_{\text{loc}}$. The latter condition implies in turn that f has the N^{-1} property (see, e.g., Theorem III.6.1 in [152]), and hence, $J_f(z)\neq 0$ a.e.; see Theorem 1 in [189]. Thus, f is a regular solution of (9.1.2). □

Corollary 9.2. *Let μ and $\nu : D\to\mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu\ (z)|<1$ a.e. and $K_{\mu,\nu}\in L^1_{\text{loc}}(D)$ and let $\psi:(0,\infty)\to(0,\infty)$ be a measurable function such that for all $0<t_1<t_2<\infty$*

$$0<\int\limits_{t_1}^{t_2}\psi(t)\,\mathrm{d}t<\infty\,,\qquad \int\limits_{0}^{t_2}\psi(t)\,\mathrm{d}t=\infty\,. \tag{9.3.5}$$

Suppose that for every $z_0\in D$ there is $\varepsilon_0<dist\,(z_0,\partial D)$ such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z)\cdot\psi^2(|z-z_0|)\,\mathrm{d}m(z)\ \le\ O\left(\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,\mathrm{d}t\right) \tag{9.3.6}$$

as $\varepsilon\to 0$. Then (9.1.2) has a regular solution.

9.4 Some Existence Theorems

Theorem 9.2. *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D\to\mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu(z)|<1$ a.e. such that*

$$K_{\mu,\nu}(z)=\frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|}\ \le\ Q(z)\ \in\ \mathrm{FMO}(D)\,. \tag{9.4.1}$$

Then the Beltrami equation (9.1.2) has a regular solution.

Proof. Lemma 9.1 yields this conclusion by choosing

$$\psi_{z_0,\varepsilon}(t)\ =\ \frac{1}{t\log\frac{1}{t}}; \tag{9.4.2}$$

see also Lemma 5.3. □

Corollary 9.3. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|}\, \mathrm{d}m(z) \ < \ \infty \qquad \forall\, z_0\in D, \tag{9.4.3}$$

then (9.1.2) has a regular solution.

Theorem 9.3. *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D\to\mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu(z)|<1$ a.e. and $K_{\mu,\nu}\in L^1_{\mathrm{loc}}(D)$. Suppose that*

$$\int\limits_0^{\delta(z_0)} \frac{\mathrm{d}r}{rk_{z_0}(r)} \ = \ \infty \qquad \forall\, z_0\in D, \tag{9.4.4}$$

where $\delta(z_0)<\mathrm{dist}\,(z_0,\partial D)$ and $k_{z_0}(r)$ is the mean of $K_{\mu,\nu}(z)$ over $|z-z_0|=r$. Then the Beltrami equation (9.1.2) has a regular solution.

Proof. Theorem 9.3 follows from Lemma 9.1 by specially choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \ \equiv \ \psi_{z_0}(t) \ : \ = \ \begin{cases} 1/[tk_{z_0}(t)]\ , & t\in(0,\varepsilon_0)\ , \\ 0\ , & \text{otherwise}, \end{cases} \tag{9.4.5}$$

where $\varepsilon_0=\delta(z_0)$. □

Corollary 9.4. *In particular, if*

$$k_{z_0}(r) \ = \ O\left(\log\frac{1}{r}\right) \quad \textit{as} \quad r\to 0 \qquad \forall\, z_0\in D, \tag{9.4.6}$$

then (9.1.2) has a regular solution.

In fact, it is clear that the condition (9.4.4) implies the whole scale of conditions in terms of log, for instance, by using functions of the form $1/(t\log\ldots\log 1/t)$.

Remark 9.1. If, in addition, the conditions (9.3.2), (9.3.6), (9.4.1), (9.4.3), (9.4.4), or (9.4.6), setting $K_{\mu,\nu}\equiv 1$ outside of D, hold at all points of ∂D in $\overline{\mathbb{C}}$, then there is a superregular solution f of (9.1.2). $K_{\mu,\nu}\equiv 1$ in $\mathbb{C}\setminus D$ means in fact that we set $\mu(z)\equiv 0\equiv\nu(z)$ in $\mathbb{C}\setminus D$. Some sense of the above conditions at ∞ if $\infty\in\partial D$ in $\overline{\mathbb{C}}$ remains to be elaborated.

The enumerated conditions for $K_{\mu,\nu}(z)$ at ∞ should be understood as the corresponding conditions for $K_{\mu,\nu}(1/\overline{z})$ at 0. Of course, all these latter conditions can also be rewritten explicitly in terms of $K_{\mu,\nu}(z)$ itself after the inverse change of variables $z\longmapsto 1/\overline{z}$. For example, the main condition (9.3.2) at ∞ has the form

$$\int\limits_{R_0<|z|<R} K_{\mu,\nu}(z)\cdot\psi^2_{\infty,R}(|z|)\,\frac{\mathrm{d}m(z)}{|z|^4} \ = \ o(I^2_\infty(R)) \qquad \text{as } R\to\infty, \tag{9.4.7}$$

where

$$0 \;<\; I_\infty(R) \;:\;=\; \int\limits_{R_0}^{R} \psi_{\infty,R}(t)\; \mathrm{d}t \;<\; \infty \tag{9.4.8}$$

for some measurable functions $\psi_{\infty,R} : (0,\infty) \to (0,\infty)$, $R \in (R_0,\infty)$.

The concept of finite mean oscillation can also be extended to infinity in the standard way. Namely, given a domain D in the extended complex plane $\overline{\mathbb{C}}$, $\infty \in D$, and a function $\varphi : D \to \mathrm{Re}$, we say that φ has *finite mean oscillation at* ∞ if the function $\varphi^*(z) = \varphi(1/\overline{z})$ has finite mean oscillation at 0. Clearly, by the inverse change of variables $z \longmapsto 1/\overline{z}$, the latter is equivalent to the condition

$$\int\limits_{|z|\geq R} |\varphi(z) - \overline{\varphi}_R|\; \frac{\mathrm{d}m(z)}{|z|^4} \;=\; O\left(\frac{1}{R^2}\right), \tag{9.4.9}$$

where

$$\overline{\varphi}_R = \frac{R^2}{\pi} \int\limits_{|z|\geq R} \varphi(z)\; \frac{\mathrm{d}m(z)}{|z|^4}\,. \tag{9.4.10}$$

Thus, to obtain a superregular solution of (9.1.2), we may assume in (9.4.1) that $Q \in \mathrm{FMO}(\overline{\mathbb{C}})$ in the given sense. In particular, the condition (9.4.3) at ∞ is rewritten in the form

$$\int\limits_{|z|>R} K_{\mu,\nu}(z)\; \frac{\mathrm{d}m(z)}{|z|^4} \;=\; O\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty\,. \tag{9.4.11}$$

The condition (9.4.4) at ∞ has the form

$$\int\limits_{\Delta}^{\infty} \frac{\mathrm{d}R}{RK(R)} \;=\; \infty \tag{9.4.12}$$

for some $\Delta > 0$, where $K(R)$ is the average of $K_{\mu,\nu}(z)$ over the circle $|z| = R$. The condition (9.4.6) at ∞ is rewritten in the form

$$K(R) \;=\; O\,(\log R) \quad \text{as} \quad R \to \infty\,. \tag{9.4.13}$$

9.5 Consequences for the Reduced Beltrami Equations

The equation of the form

$$f_{\overline{z}} = \lambda(z)\; \mathrm{Re}\, f_z \tag{9.5.1}$$

with $|\lambda(z)| < 1$ a.e. is called the *reduced Beltrami equation*; see, e.g., [26, 44–47, 265]. Equation (9.5.1) can be rewritten as (9.1.2) with

$$\mu(z) \;=\; \nu(z) \;=\; \frac{\lambda(z)}{2} \tag{9.5.2}$$

and then

$$K_{\mu,\nu}(z) \;=\; K_\lambda(z) \;:\;=\; \frac{1+|\lambda(z)|}{1-|\lambda(z)|}\,. \tag{9.5.3}$$

Thus, the results of the previous sections give the following consequences for the reduced Beltrami equation (9.5.1):

Theorem 9.4. *Let D be a domain in $\mathbb{C}$ and let $\lambda : D \to \mathbb{C}$ be a measurable function with $|\lambda(z)| < 1$ a.e. such that*

$$K_\lambda(z) = \frac{1+|\lambda(z)|}{1-|\lambda(z)|} \;\le\; Q(z) \;\in\; \mathrm{FMO}(D). \tag{9.5.4}$$

Then the Beltrami equation (9.5.1) has a regular solution.

Corollary 9.5. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \;\; \fint_{B(z_0,\varepsilon)} \frac{1+|\lambda(z)|}{1-|\lambda(z)|}\,\mathrm{d}m(z) \;<\; \infty \qquad \forall\, z_0 \in D, \tag{9.5.5}$$

then (9.5.1) has a regular solution.

Theorem 9.5. *Let D be a domain in $\mathbb{C}$ and let $\lambda : D \to \mathbb{C}$ be a measurable function with $|\lambda(z)| < 1$ a.e. and $K_\lambda \in L^1_{\mathrm{loc}}(D)$. Suppose that*

$$\int\limits_0^{\delta(z_0)} \frac{\mathrm{d}r}{r\Lambda_{z_0}(r)} \;=\; \infty \qquad \forall\, z_0 \in D, \tag{9.5.6}$$

where $\delta(z_0) < \mathrm{dist}\,(z_0, \partial D)$ and $\Lambda_{z_0}(r)$ is the mean of $K_\lambda(z)$ over $|z - z_0| = r$. Then the Beltrami equation (9.5.1) has a regular solution.

Corollary 9.6. *In particular, if*

$$\Lambda_{z_0}(r) \;=\; O\left(\log\frac{1}{r}\right) \quad \textit{as} \quad r \to 0 \qquad \forall\, z_0 \in D, \tag{9.5.7}$$

then (9.5.1) has a regular solution.

Remark 9.2. If, in addition, the conditions (9.5.4)–(9.5.7), setting $K_l \equiv 1$ outside of D, hold at all points of ∂D in $\overline{\mathbb{C}}$, then there is a superregular solution f of (9.5.1);

$K_l \equiv 1$ in $\mathbb{C}\setminus D$ means that we set $\lambda(z) \equiv 0$ in $\mathbb{C}\setminus D$. For a sense of these conditions at ∞ if $\infty \in \partial D$ in $\overline{\mathbb{C}}$, see Remark 9.1. In particular, the reduced Beltrami equation (9.5.1) has a superregular solution if a.e. in D

$$K_\lambda(z) = \frac{1+|\lambda(z)|}{1-|\lambda(z)|} \leq Q(z) \in \mathrm{BMO}(\mathbb{C}); \tag{9.5.8}$$

see Theorem 1.2 in [48]. All the above results remain true for the case in (9.1.2) when

$$\nu(z) = \mu(z)\, \mathrm{e}^{\mathrm{i}\theta(z)} \tag{9.5.9}$$

with an arbitrary measurable function $\theta(z) : D \to \mathrm{Re}$ and, in particular, for equations of the form

$$f_{\bar{z}} = \lambda(z)\, \mathrm{Im}\, f_z \tag{9.5.10}$$

with a measurable coefficient $\lambda : D \to \mathbb{C}$, $|\lambda(z)| < 1$ a.e.; see, e.g., [44–47].

9.6 Existence Theorems with Integral Conditions

We obtain the next significant result immediately from Theorem 9.3 and Corollary 2.7.

Theorem 9.6. *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu(z)| < 1$ a.e. such that*

$$\int_D \Phi(K_{\mu,\nu}(z))\, \mathrm{d}m(z) < \infty, \tag{9.6.1}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function. If Φ satisfies at least one of the conditions (2.8.12)–(2.8.17), then the Beltrami equation (9.1.2) has a regular solution.

Remark 9.3. The condition (9.6.1) can be also localized to neighborhoods U_{z_0} of points $z_0 \in D$ with $\Phi = \Phi_{z_0}$ under the same conditions on the functions Φ_{z_0}. If $\infty \in D$, then the condition (9.6.1) for $K_{\mu,\nu}(z)$ at $\infty \in D$ should be understood as the corresponding condition for $K_{\mu,\nu}(1/\bar{z})$ at 0. The latter condition can also be rewritten explicitly in terms of $K_{\mu,\nu}(z)$ itself, after the inverse change of variables $z \longmapsto 1/\bar{z}$, in the form

$$\int_{U_\infty} \Phi_\infty(K_{\mu,\nu}(z))\, \frac{\mathrm{d}m(z)}{|z|^4} < \infty\,. \tag{9.6.2}$$

If the domain D is unbounded, then it is better to use the global condition

$$\int_D \Phi(K_{\mu,\nu}(z)) \, \frac{\mathrm{d}m(z)}{(1+|z|^2)^2} \; < \; \infty \tag{9.6.3}$$

instead of the condition (9.6.1). The latter means integration of the function $\Phi \circ K_{\mu,\nu}$ in the spherical area.

We may assume in the above theorem that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex and nondecreasing on the whole segment $[0,\infty]$, but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$. Indeed, every function $\Phi : [0,\infty] \to [0,\infty]$ which is convex and nondecreasing on a segment $[T,\infty]$, $T \in (0,\infty)$, can be replaced by a nondecreasing convex function $\Phi_T : [0,\infty] \to [0,\infty]$ in the following way: We set $\Phi_T(t) \equiv 0$ for all $t \in [0,T]$, $\Phi(t) = \varphi(t)$, $t \in [T,T_*]$, and $\Phi_T \equiv \Phi(t)$, $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,T)$ and touching the graph of the function $\tau = \Phi(t)$ at a point $(T_*,\Phi(T_*))$, $T_* \geq T$. For such a function, we have by the construction that $\Phi_T(t) \leq \Phi(t)$ for all $t \in [1,\infty]$ and $\Phi_T(t) = \Phi(t)$ for all $t \geq T_*$.

From Theorem 9.6, we obtain the following consequence for the reduced Beltrami equations (9.5.1):

Theorem 9.7. *Let D be a domain in $\mathbb{C}$ and let λ be a measurable function with $|\lambda(z)| < 1$ a.e. such that*

$$\int_D \Phi(K_\lambda(z)) \, \mathrm{d}m(z) \; < \; \infty, \tag{9.6.4}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a nondecreasing convex function. If Φ satisfies at least one of the conditions (2.8.12)–(2.8.17), then the reduced Beltrami equation (9.5.1) has a regular solution.

Remark 9.4. Remark 9.3 are valid for the reduced Beltrami equation. Moreover, the above results remain true for the case in (9.1.2) when

$$\nu(z) \; = \; \mu(z)\, \mathrm{e}^{\mathrm{i}\theta(z)} \tag{9.6.5}$$

with an arbitrary measurable function $\theta(z) : D \to \mathbb{R}$ and, in particular, for equations of the form

$$f_{\bar{z}} = \lambda(z) \, \mathrm{Im}\, f_z \tag{9.6.6}$$

with a measurable coefficient $\lambda : D \to \mathbb{C}$, $|\lambda(z)| < 1$ a.e.; see, e.g., [44–46].

Next, note that the example following the proof of Theorem 7.8 (see Lemma 7.2) from a previous chapter for the Beltrami equations of the first type (B) shows that the conditions (2.8.12)–(2.8.17) are not only sufficient but also necessary for the Beltrami equations with two characteristics (9.1.2) to have regular solutions.

Finally, the same is valid for the reduced Beltrami equations (9.5.1) because the examples in the mentioned theorem had the form

$$f(z) \;=\; \frac{z}{|z|}\,\rho(|z|),$$

where $\rho(t) = \mathrm{e}^{I(t)}$ and

$$I(t) \;:\; = \int\limits_0^t \frac{\mathrm{d}r}{rK(r)}\ .$$

Indeed, setting $z = r\mathrm{e}^{\mathrm{i}\vartheta}$, we have by (7.5.10) and (7.5.11) that

$$\frac{\partial f}{\partial z} \;=\; \frac{1}{2}\left(\frac{\rho(r)}{rK(r)} + \frac{\rho(r)}{r}\right) = \frac{\rho(r)}{2r}\cdot\frac{1+K(r)}{K(r)} \;>\; 0$$

and

$$\frac{\partial f}{\partial \overline{z}} \;=\; \frac{\mathrm{e}^{2\mathrm{i}\vartheta}}{2}\left(\frac{\rho(r)}{rK(r)} - \frac{\rho(r)}{r}\right) = \mathrm{e}^{2\mathrm{i}\vartheta}\cdot\frac{\rho(r)}{2r}\cdot\frac{1-K(r)}{K(r)}\ ,$$

i.e.,

$$\lambda(z) \;=\; \mathrm{e}^{2\mathrm{i}\vartheta}\cdot\frac{1-K(r)}{1+K(r)} \;=\; -\,\frac{z}{\overline{z}}\cdot\frac{K(|z|)-1}{K(|z|)+1}$$

and, consequently, $K_\lambda(z) \;=\; K(|z|)$.

Chapter 10
Alternating Beltrami Equation

10.1 Introduction

Roughly speaking, this term refers to equation (B) in the case where μ is a measurable complex-valued function in a domain D in $\mathbb{C}$ with $|\mu| < 1$ a.e. in parts of D and $|\mu| > 1$ a.e. in other parts of D. With further assumptions on μ, every nonconstant solution $f : D \to \mathbb{C}$ of (B), i.e., an ACL mapping which satisfies (B) a.e., is locally quasiregular (and in particular open, discrete, and sense preserving) in the regions where $|\mu| < 1$ and anti-quasiregular (and in particular open, discrete, and sense-reversing) in the regions where $|\mu| > 1$. These mappings may branch at some isolated points in D or may have folds and cusps. This leads to the notation and study of, what we call, *folded quasiregular maps* (abbreviated as FQR-maps) and *branched folded maps* (abbreviated as BF-maps).

A first systematic study of alternating Beltrami equations which started in [233] and continued in [235, 236] revealed new phenomena, which may have applications in other areas. Earlier results in this subject can be found in [1, 75, 266, 271].

10.1.1 The Geometric Configuration and the Conditions on μ

Let E be a one-dimensional set in D of two-dimensional Lebesgue measure zero, such that (D,E) is a locally finite two-color map painted black and white; see Fig. 4. The local assumption on E says that every point z of E has a neighborhood U such that $E \cap U$ is an arc (see Fig. 5) or a (finite) *star* with a *vertex* at z, i.e., a union of finitely many arcs, each connecting the point z with ∂U and which are disjoint except for their common end point z; see Fig. 6. Below, we will assume that $\mu : D \to \overline{\mathbb{C}}$ is a measurable function such that $|\mu| < 1$ a.e. in the white regions of $D \backslash E$ and $1/|\mu| < 1$ a.e. in the black regions of $D \backslash E$.

V. Gutlyanskii et al., *The Beltrami Equation: A Geometric Approach*, Developments in Mathematics 26, DOI 10.1007/978-1-4614-3191-6_10,

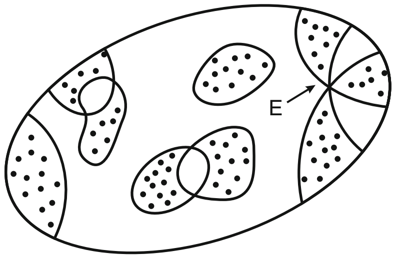

Fig. 4 A two-color map

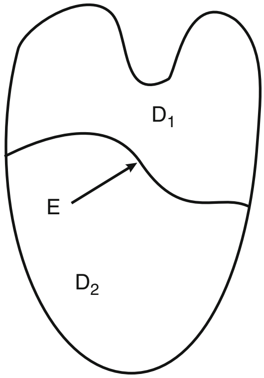

Fig. 5 A cross-cut

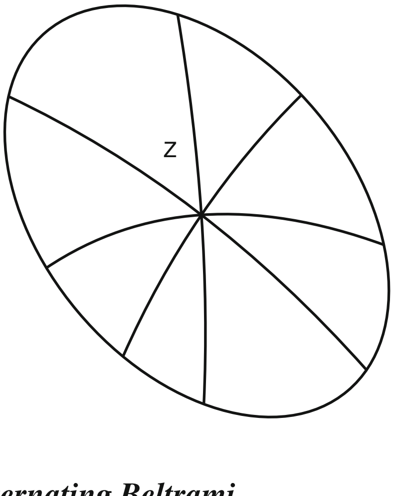

Fig. 6 A star

10.1.2 A Symmetric Form of the Alternating Beltrami Equation

In order to include the case $w_z = 0$, we rewrite (B) in a symmetric form:

$$A(z)w_z + B(z)w_{\bar{z}} = 0, \tag{10.1.1}$$

where $A(z)$ and $B(z)$ are measurable functions in D and $\mu = -A/B$ at points where $B \neq 0$. Then $|A(z)| < |B(z)|$ a.e. in the white regions and $|A(z)| > |B(z)|$ a.e. in the black regions. With no loss of generality, we may use (B) instead of (10.1.1), agreeing that $\mu = \infty$ corresponds to $B = 0$ in (10.1.1). We will assume that at every point of $D \backslash E$ either $|A/B|$ or $|B/A|$ is locally bounded, implying that (10.1.1) is locally strongly elliptic in $D \backslash E$. Note that A, B, and μ may be continuous on E. We will be mainly interested in solutions which are branched folded mappings in D. These mappings are described in the following sections.

10.1.3 Branched Folded Maps

The notion of a *branched folded covering* (abbreviated as BF-covering) was introduced by Tucker [253]. However, the class of BF-maps and related classes of maps were studied earlier by Hopf [109] and others, and in the last 30 years by Church [62], Church and Timourian [64], Väisälä [259], and others. More recently, these mappings have appeared in connection with problems in graph theory (cf. [52]), in the study of light harmonic functions [156], in the study of nonlinear operators in Banach spaces and elliptic boundary value problems [39, 63], and in some aspects of singularity theory [20]. Here, BF-maps are defined in a wider sense, which is more appropriate for the study of alternating Beltrami equations where smoothness is not assumed.

10.1.4 Definition of BF-Maps

Let X and Y be oriented surfaces. A map $f : X \to Y$ is called a *branched folded map* (abbreviated as BF-map) if f is *discrete*, i.e., the points of $f^{-1}(y)$ are isolated in X for any y in Y, and if there exists a one-dimensional set E in X such that (X, E) is a locally finite two-color map painted black and white such that f is open and sense preserving in the white regions and open and sense-reversing in the black regions.

BF-maps according to the classical definition, simplicial maps, open discrete maps, nonconstant quasiregular maps, and nonconstant holomorphic maps form subclasses of BF-maps. A BF-map in the classical definition is locally simplicial, and its critical set looks like the critical set of a BF-map in our definition. The critical set of a classical BF-map and its image look alike. This need not be the case for a BF-map in our definition.

10.1.5 Discrete Maps, Canonical Maps, and Classification of Critical Points

Let $f : X \to Y$ be a discrete map, where X and Y are oriented surfaces, and let Σ_f denote the *critical set* of f, i.e., the set of points where f is not a local homeomorphism. The critical points of f, i.e., points of Σ_f, are classified into three categories: *branch points, folding points*, and *cusp points*.

A critical point is called a *branch point* if it is isolated in Σ_f. A critical point p will be called a *folding point* if p has a neighborhood U such that $f|U$ is topologically equivalent to the *standard folding* map $(x,y) \to (x,|y|)$ of R^2 into itself, where p corresponds to the point $(0,0)$. A critical point which is neither a branch point nor a folding point will be called a *cusp point*. The behavior of a map near a cusp point is not well understood. Only some special cusp points will be classified and described here.

10.1.6 Branch Points, Power Maps, and Winding Maps

If p is a branch point for a discrete map $f : X \to Y$, where X and Y are as above, then p has a neighborhood in which f is topologically equivalent to a power map $z \to z^d$ of $\mathbb{C}$ for some integer $d = d(p) > 1$ where the point p corresponds to the point $z = 0$ in the map z^d. This result follows from the fact that f must be open in some neighborhood of p (cf. [168], Theorem 2.10) and by a theorem of Stoilow (cf. [240, 269]). Note that the power map $z \to z^d$ is topologically equivalent to a *winding map* g_d *of order* d, which is defined by

$$g_d(r\mathrm{e}^{\mathrm{i}t}) = r\mathrm{e}^{\mathrm{i}dt}, \quad d \in Z_+.$$

10.1.7 Folding Maps

A map $f : X \to Y$, X and Y as above, is a *proper folding* if f is topologically equivalent to the standard folding $(x,y) \to (x,|y|)$ of R^2. In this case, X and Y must be simply connected, the critical set Σ_f is a line, called a *folding line*, and every point of Σ_f is a folding point. A *folding map* is defined as a restriction of a proper folding map g to a domain, say U, such that $U \cap \Sigma_g$ is connected.

One may consider folding maps in the following wider sense: $f : X \to Y$ is a *folding map* along a *folding line* l, if l is a Jordan arc or a Jordan curve in X, $X \setminus l$ has two components X_1 and X_2, $f|X_1 \cup l$ is a sense-preserving embedding and $f|X_2 \cup l$ is a sense-reversing embedding. If $X \subset \mathbb{C}$ and $Y = \bar{\mathbb{C}}$, the definitions agree.

10.1.8 (p,q)-Cusp Maps and (p,q)-Cusp Points

The simplest cusp point appears as the point $(0,0)$ in *Whitney's canonical cusp mapping*

$$(x,y) \to (x, xy - y^3).$$

This map is topologically equivalent to the so-called *straight (3,1)-cusp map* $f_{3,1}$ of $\mathbb{C}$ onto itself which is defined by $f_{3,1}(z) = z^3/|z|^2$ for Im $z > 0$ and $f_{3,1}(z) = \bar{z}$ for Im $z < 0$. Similarly, the *straight* (p,q)*-cusp map* $f_{p,q}$ of $\mathbb{C}$ is defined for any positive integers p and q with $p \equiv q$ $(mod\ 2)$ by $z^p/|z|^{p-1}$ if Im $z > 0$, and $\bar{z}^q/|z|^{q-1}$ if Im $z < 0$. Then the critical set of $f_{p,q}$ is the real axis R, the point $z = 0$ is a (p,q)-*cusp point*, and every other point on the real axis is a folding point. Also, $f_{p,q}(x) = x$ for $x \in R$ if p and q are odd and $f_{p,q}(x) = |x|$ for $x \in R$ if p and q are even. A map $f : D \to \mathbb{C}$ is called a *proper* (p,q)-cusp map if it is topologically equivalent to the straight (p,q)-cusp map, and the point p in D which corresponds to $z = 0$ is a (p,q)-cusp point. The restriction of f to a subdomain of D which contains the cusp point is called a (p,q)-cusp map. Some of these (p,q)-cusp maps are considered in Lyzzaik [156] in connection with the local behavior of light harmonic maps.

10.1.9 Umbrella and Simple Umbrella Maps

Next, we describe a wider class of cusp points. These are associated with what are called *umbrella maps* and *simple umbrella maps.* Mappings of this type are considered in [123, 156]. A map $f : \mathbb{C} \to \mathbb{C}$ is called a *simple straight umbrella map* if $f(0) = 0$ and $f(z) = |z|\,\varphi(z/|z|)$ for $z \neq 0$, for some continuous piecewise injective function $\varphi : S^1 \to S^1$, where S^1 is the unit circle. A mapping $f : X \to Y$ is called a *proper simple umbrella map,* if f is topologically equivalent to a simple straight umbrella map. The restriction of a proper simple umbrella map $f : X \to Y$ to a domain D in X is called a *simple umbrella map*.

A mapping $f : \mathbb{C} \to \mathbb{C}$ is called a *straight umbrella map* if $f(0) = 0$ and $f(z) = t\gamma_t(z/|z|)$ for $|z| = t > 0$ for some path $\gamma : [0,\infty) \to F_d$ in the space F_d of all continuous and piecewise injective maps of S^1 into itself of a given degree d, which has the property that all maps $\gamma_t = \gamma(t) : S^1 \to S^1,\ \ t \in [0,\infty)$ have the same critical points on S^1. A mapping $f : X \to Y$ is called a *proper umbrella map*, if f is topologically equivalent to a straight umbrella map. The restriction of f to a domain in X is called an *umbrella map.*

Note that the critical set Σ_f of a proper umbrella map f is either empty, consists of a single point, or is a star. In the latter case, the vertex of the star will be called also the *vertex of the umbrella map*, and it is then either a folding point or a cusp point. Also, note that Σ_f may be a star while its image need not be a star. If, however, f is a proper simple umbrella map, and if its critical set is a star, so is its image.

One can show that proper umbrella maps and proper (p,q)-cusp maps are proper in the usual sense and that the restriction of umbrella, resp. simple umbrella maps to certain subdomains are proper umbrella, resp. simple umbrella, maps.

Locally, a discrete map or even a BF-map $f : D \to \mathbb{C}$ need not be equivalent to any of the canonical maps mentioned above. However, one can see that if E is a free arc in the critical set Σ_f of a BF-map $f : D \to \mathbb{C}$, and if $f(E)$ is contained in an arc, then every point z of E has a neighborhood U such that $U \subset D$ and such that $f|U$ is either a proper folding map or a proper (p,q)-cusp map for some odd integers p and q which depend on the point z. As for the local behavior of a BF-map $f : D \to \mathbb{C}$ at a vertex z_0 of a star E in the critical set Σ_f of f, one can show that if $f(E)$ is included in a star and the image of each arm of E is included in an arm of $f(E)$ and if $f|E$ is piecewise injective, then the point z_0 has a neighborhood U such that $f|U$ is a proper simple umbrella map.

10.2 Branched Folded Maps

The main purpose of this section is to study the local behavior of a BF-map at a critical point. In general, a BF-map need not behave at a critical point like any of the canonical maps, which we described in Sects. 10.1.6–10.1.9. However, we will show that under some assumptions on the image of the critical set, a BF-map does behave like one of the canonical maps at every critical point. It should be mentioned here that at an isolated critical point, a BF-map behaves like a power map or, equivalently, like a winding map. Indeed, one can show that a discrete map is open in a neighborhood of an isolated critical point (see [168], Theorem 2.10), and then the result follows by a theorem of Stoilow; cf. [240, 269]. For convenience, we quote this result in the following lemma:

Lemma 10.1. *Let $f : D \to \mathbb{C}$ be a discrete map which is open in $D \backslash \{a\}$, for some point a in D with $f(a) = 0$. Then the point a has a simply connected neighborhood U and a homeomorphism h of D onto a disc $D_r = \{z : |z| < r\}$ such that $h(a) = 0$ and such that $f|U = g_p \circ h$ for some integer $p = p(a) > 0$, where $g_p(\rho \mathrm{e}^{\mathrm{i}t}) = \rho \mathrm{e}^{\mathrm{i}pt}$ is a winding map. Furthermore, h is sense preserving if f is sense preserving.*

We will conclude this section with examples of BF-maps which do not behave locally like any of the canonical maps and with examples of discrete maps which differ completely from BF-maps in terms of the shape of the critical set.

Proper folding maps, proper (p,q)-cusp maps, and proper umbrella maps were defined in 10.1.6–10.1.9 as being topologically equivalent to the standard folding map $(x,y) \to (x,|y|)$ of R^2, to the straight (p,q)-cusp map, and to the straight umbrella map, respectively. Each of the latter maps is proper, in the sense that the preimage of every compact set is compact; therefore, proper folding maps, proper (p,q)-cusp maps, and proper umbrella maps are proper in the usual sense. Since we are mostly interested here in the local properties of BF-maps, the following two lemmas are relevant to our study:

Lemma 10.2. *If f is a folding map of a domain D with a folding line $E, E \subset D$, then E has a simply connected neighborhood U such that $U \subset D$ and such that $f|U$ is a proper folding map.*

Proof. By definition, f is a restriction of a proper folding map which, in turn, is topologically equivalent to the standard folding map $(x,y) \to (x,|y|)$ of R^2. Hence, it suffices to prove the lemma in case that E is an arc in the real axis R, $E \subset D$ and $f(x,y) = (x,|y|)$ for $(x,y) \in D$. In this situation, the statement is clear, and thus, the general case follows. □

Lemma 10.3. *Let f be a (p,q)-cusp map or an umbrella map of a domain D with a cusp point at a point $a \in D$. Then the point a has a neighborhood U such that $U \subset D$ and such that $f|U$ is a proper (p,q)-cusp map or a proper umbrella map, respectively.*

Proof. As in Lemma 10.2. □

Next, we study the local behavior of a BF-map f at a free arc E in the critical set of f when $f(E)$ is again an arc. Since we are interested merely in the local behavior of f, we will assume that f is defined in a simply connected neighborhood of E.

Theorem 10.1. *Let D be a simply connected domain in $\mathbb{C}$, E a crosscut in D, i.e., an arc in D with end points on ∂D, D_1 and D_2 the two connected components of $D \backslash E$ (see Fig. 5), $f : D \to \mathbb{C}$ a discrete map, which is open and sense preserving in D_1, and open and sense-reversing in D_2 such that $f(E)$ is an arc.*

Then every point a in E has a neighborhood U such that $U \subset D$ and such that $f|U$ is either a proper folding map or a proper (p,q)-cusp map with a cusp point at the point a for some positive integers p and q with $p \equiv q (mod\, 2)$, which depend on the point a.

Furthermore, if, in addition, $f|E$ is injective, then all cusps that f has on E are odd.

Proof. Since in R^2 every arc is tame (cf. [175], p. 72), we may assume that E and $f(E)$ are subarcs of the real axis R. We may also assume that D is symmetric with respect to R.

Let f_1 denote the symmetric extension of $f|D_1 \cup E$, i.e., $f_1(z) = f(z)$ for $z \in D_1 \cup E$ and $f_1(z) = \bar{f}(\bar{z})$ for $z \in D_2$. Then f_1 is continuous and discrete in D, open and sense preserving in $D_1 \cup D_2 = D \backslash E$, and $f_1(E) \subset R$.

It then follows by [252], Theorem B, that f_1 is open in D. Then, in view of Stoilow's theorem and the topological equivalence between the power maps z^d and the winding map g_d, it follows that every point a in E has a neighborhood V_1 and sense-preserving homeomorphism h_1 of V_1 onto a disc $D_{r_1} = \{z : |z| < r_1\}$ for some $r_1 > 0$, with $h_1(a) = 0$, and a positive integer p such that $V_1 \subset D$ and such that $f_1(z) = g_p(h_1(z)) + f_1(a)$ for all $z \in V_1$, where $g_p(z)$ is a winding map of order p.

In a similar way, a symmetric extension of $f|D_2 \cup E$ yields a discrete map $f_2 : D \to \mathbb{C}$, with $f_2(z) = f(z)$ for $z \in D_2 \cup E$ and $f_2(z) = \bar{f}(\bar{z})$ for $z \in D_1$, which is open and sense-reversing in $D \backslash E$. Again by [252], f_2 is open, and by Stoilow's theorem every point a in E has a neighborhood V_2 and a sense-preserving homeomorphism

h_2 of V_2 onto a disc $D_{r_2} = \{z : |z| < r_2\}$ for some $r_2 > 0$, with $h_2(a) = 0$, and positive integer q such that $V_2 \subset D$ and such that $f_2(z) = g_{-q}(h_2(z)) + f_2(a)$ for all $z \in V_2$, where $g_{-q}(z)$ is a winding map of order $-q$.

Since $g_p(e^{2k\pi i/p}z) = g_p(z)$ for $k = 1, 2, \ldots, p-1$, and $g_{-q}(e^{2m\pi i/q}z) = g_{-q}(z)$ for $m = 1, 2, \ldots, q-1$, we may replace h_1 and h_2 by the map $z \to e^{2k\pi i/p}h_1(z)$ and by the map $z \to e^{2m\pi i/q}h_2(z)$, respectively, for some k and m, and thus may assume that each of the maps h_1 and h_2 maps $E \cap V_1 \cap V_2$ into R, preserving orientation.

Now, f_1 and f_2 agree on $E \cap V_1 \cap V_2$ and g_p and g_{-q} agree on $f_i(E \cap V_1 \cap V_2) = f(E \cap V_1 \cap V_2), i = 1, 2$. Therefore $p \equiv q (\text{mod } 2)$, the map h, which is defined as $h = h_1$ on $\bar{D}_1 \cap V_1 \cap V_2$ and $h = h_2$ on $\bar{D}_2 \cap V_1 \cap V_2$, is a homeomorphism of $V = V_1 \cap V_2$ onto a disc $D_r = \{z : |z| < r\}, r = \min(r_1, r_2)$, the map g which is defined as g_p in the upper half of D_r and g_{-q} in the lower half of D_r is a proper folding if $p = q = 1$, and it is a proper (p,q)-cusp map otherwise, and $f(z) = g(h(z)) + f(a)$ for all $z \in V$.

This shows that $f|V$ is a proper folding or a proper (p,q)-cusp map. Finally, if p and q are even at a cusp point a on E, then $f|E$ is not injective in any neighborhood of a. This shows the only way that $f|E$ is injective is if all cusp points of f on E are odd. □

Corollary 10.1. *Let E be a Jordan curve in S^2, D_1 and D_2 the two connected components of $S^2 \backslash E$, $f : S^2 \to S^2$ a discrete map, which is open and sense preserving in D_1, and open and sense-reversing in D_2, such that $f(E)$ is an arc or a Jordan curve.*

Then every point a of E has a neighborhood V such that $f|V$ is either a folding map or a (p,q)-cusp map.

Furthermore:

(i) f has at most finitely many cusp points on E.
(ii) If $f|E$ is injective, then all cusp points on E are odd.
(iii) If $f(E)$ is an arc, then f has at least two even cusp points on E.

Sections 10.1.1 and 10.1.2 concern two classes of maps, defined in domains in $\mathbb{C}$ and in S^2, respectively. A straight (p,q)-cusp map belongs to the first class. It has only one cusp point in $\mathbb{C}$, and if p and q are odd, it maps R homeomorphically onto itself. In this case, this map has a continuous extension to $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \bar{\mathbb{C}}$ being homeomorphic to S^2.

The extended map is discrete in $\bar{\mathbb{C}}$, open and sense preserving in the upper half plane, open and sense-reversing in the lower half plane, it maps $\bar{R} = R \cup \{\infty\}$, which is a Jordan curve in $\bar{\mathbb{C}}$, homeomorphically (in fact by identity) onto itself, and it has two cusp points (of the same orders (p,q)), one at $z = 0$ and one at ∞.

Similarly, an even straight (p,q)-cusp map of $\mathbb{C}$ extends to a BF-map of $\bar{\mathbb{C}}$ with two cusp points, 0 and ∞.

One may ask whether a map described in Theorem 10.1 may have more than two cusp points in $\bar{\mathbb{C}}$. The following proposition not only gives an affirmative answer to the question but also shows that one can prescribe the cusp points a_k and arbitrary odd orders (p_k, q_k) at each cusp point a_k. We do not know if even orders can be prescribed arbitrarily in the same way.

Proposition 10.1. *Let x_k, $k = 1,2,\ldots$ be points on the real axis R with no accumulation point in R and with $|x_1| < |x_2| < \cdots$, (n_k, m_k), $k = 1,2,\ldots$ be pairs of nonnegative integers such that $n_k + m_k > 0$. Then there exists a BF-map F in R^2 whose critical set is the set R, such that every point of R except for the points x_k, $k = 1,2,\ldots$, is a folding point of F and such that x_k is a cusp point of order $(2n_k+1, 2m_k+1)$, $k = 1,2,\ldots$.*

Proof. It is easy to see that the function

$$f_1(z) = \Pi_{k=1}^{\infty}(1-z/x_k)^{2n_k}\mathrm{e}^{2n_k(z/x_k+\cdots+(z/x_k)^k/k)}$$

if $x_1 \neq 0$, and

$$f_1(z) = z^{2n_1}\Pi_{k=2}^{\infty}(1-z/x_k)^{2n_k}\mathrm{e}^{2n_k(z/x_k+\cdots+(z/x_k)^k/k)}$$

if $x_1 = 0$, is an entire function with zeros at $z = x_k$ of the orders $2n_k$, $k = 1,2,\ldots$. Furthermore, $f_1|R$ is real-valued and non-negative.

Then

$$F_1(z) = \int_0^z f_1(z)\mathrm{d}z$$

is an entire function, $F_1|R$ is a strictly increasing real-valued function, and $F_1(z)$ is locally topologically equivalent to $z \to (z-x_k)^{2n_k+1}$ at the point $z = x_k$.

Similarly, there is an entire function $F_2(z)$ which is topologically equivalent to $z \to (z-x_k)^{2m_k+1}$ at the point $z = x_k$, and such that $F_2|R$ is a strictly increasing real-valued function.

Define the function $\phi : R \to R$ by $\phi : x \to F_1 \circ F_2^{-1}(x)$. Then ϕ is an automorphism of R.

The function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ which is defined by $\Phi(u,v) = (\phi(u), -v)$ is an automorphism of $\mathbb{R}^2$. Then the mapping F which is given by $F(z) = F_1(z)$ if $\operatorname{Im} z \geq 0$ and $F(z) = \Phi \circ F_2(z)$ if $\operatorname{Im} z < 0$ satisfies the requirements of the proposition.

We now turn to the local behavior of a BF-map f at a vertex of a star in the critical set of f. We may restrict ourselves to a simply connected neighborhood of the vertex. □

Theorem 10.2. *Let D be a simply connected domain in $\mathbb{C}$, E a star in D with a vertex at a point a, $a \in D$ and arms $l_1, l_2, \ldots, l_{2n}$, each of which has one end point at the point a and one end on ∂D (see Fig. 3).*

Let $D_1, D_2, \ldots, D_{2n}$ denote the connected components of $D \backslash E$ such that $\bar{D}_{i-1} \cap \bar{D}_i = l_i$ for $i = 1,2,\ldots,2n$ and $\bar{D}_1 \cap \bar{D}_{2n} = l_1$.

Let $f : D \to \mathbb{C}$ be a discrete map, which is open and sense preserving in the domains $D_2, D_4, \ldots, D_{2n}$, and open and sense-reversing in D_1, $D_3, \ldots, D_{2n-1}$, such that $f(E)$ is included in star E' with a vertex at the point a', such that $f(a) = a'$, and such that $f(l_i), i = 1,\ldots,2n$ is included in an arm of E'.

Then the point a has a neighborhood W such that $W \subset D$ and such that $f|W$ is a proper simple umbrella map.

Proof. Since every star in $\mathbb{R}^2$ can be tamely embedded in a straight star (cf. [175], p. 76), we may assume that $D = \mathbb{R}^2$, that E is a straight star with a vertex at $z = 0$, whose arms $l_1, l_2, \ldots, l_{2n}$ are in directions $0, \pi/n, \ldots, (2n-1)\pi/n$, respectively, and that $f(E)$ is included in a straight star with a vertex at the point $z = 0$, and that $f(0) = 0$. With no loss of generality, we may assume that $f(l_1) = l_1$.

We consider separately the two cases:

Case 1. $f(l_2) = f(l_1) = l_1$.

Case 2. $f(l_2) \neq f(l_1)$.

In the latter case, $\arg z = \beta$ for $z \in f(l_2)$ for some $\beta \in (0, 2\pi)$.

Let ψ be an automorphism of $\mathbb{C}$, which is the identity in Case 1, and which in Case 2 is a piecewise winding map which maps the sector $S = \{z : 0 < \arg z < \beta\}$ onto the upper half plane $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and maps the complementary sector $\mathbb{C} \backslash S$ onto $\mathbb{C} \backslash U$;

$$\psi(re^{it}) = re^{i\pi t/\beta} \quad \text{for } 0 < t < \beta,$$

and

$$\psi(re^{it}) = re^{i[\pi + \pi(t-\beta)/(2\pi-\beta)]} \quad \text{for } \beta \le t \le 2\pi.$$

In either case

$$\psi(f(\partial D_1)) = \psi(f(l_1) \cup \psi(f(l_2) \subset R.$$

Next, let $\phi = g_n|\bar{D}_1$ and $F = \psi \circ (f(\bar{D}_1) \circ \phi^{-1}$. Then ϕ is a homeomorphic winding map of D_1 onto $\bar{U}$, and F is a map of $\bar{U}$ into $\mathbb{C}$, which is discrete in $\mathbb{C}$, open and sense-reversing in U, and maps R into itself. The symmetric extension of F of $\mathbb{C}$, which is denoted again by F, and defined by $F(z) = \bar{F}(\bar{z})$, is discrete in $\mathbb{C}$, open and sense-reversing in $\mathbb{C} \backslash R$ and maps R into itself. Then, by [252], Theorem B, F is open in $\mathbb{C}$. Therefore, by Stoilow's theorem, there exist a neighborhood V of the point $z = 0$ and sense-preserving homeomorphism h of V onto a disc $D_r = \{z \in \mathbb{C} : |z| < r\}$ such that $F|V = g_{-p} \circ h$ for some positive integer p.

Let $W_1 = \phi^{-1}(V \cap \bar{U})$. Then $W_1 \subset \bar{D}_1$, and W_1 is a neighborhood of the point $z = 0$ relative to $\bar{D}_1$, and $\Phi_1 = g_{1/n} \circ h \circ (\phi|W_1)$ maps W_1 homeomorphically, and in a sense-preserving way, onto the region

$$W_1' = \{z \in \mathbb{C} : 0 \le \arg z \le \pi/n,\ |z| < r_1\},$$

such that $\Phi_1(0) = 0$. The map $g_{-p} \circ (g_n|W_1')$ is a sense-reversing winding map of W_1' into D_{r_1}, and $G_1 = \psi \circ g_{-p} \circ (d_n|W_1'))$ is a piecewise winding map of W_1' into $\mathbb{C}$ with $G_1(0) = 0$.

In the same way, we can construct for each sector D_k, $k = 1, 2, \ldots, 2n$, a region W_k, with $W_k \subset D_k$, a sense-preserving homeomorphism Φ_k of W_k onto the region

$$W_k' = \{z \in \mathbb{C} : (k-1)\pi/n \le \arg z \le k\pi/n, |z| < r_k\}$$

and a piecewise winding map Ψ_k of W_k' into $\mathbb{C}$, such that $f|W_k = \Psi_k \circ \Phi_k$.

By restricting ourselves to a subregion of W_k, if needed, for instance by redefining $W_k = \Phi_k(D_r)$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $r = \min\{r_k, k = 1,2,\ldots,2n\}$, we may assume that $r_1 = r_2 = \cdots = r_{2n} = r > 0$.

We now show that $\Phi_{k-1}(z) = \Phi_k(z)$ for all $z \in W_{k-1} \cap W_k \subset l_1$ for $k = 2,3,\ldots,2n$, with a similar relation for $z \in W_{2n} \cap W_1 \subset l_k$.

Let $z \in W_{k-1} \bigcap W_k$, then arg $\Phi_{k-1}(z) =$ arg $\Phi_k(z)$. Next, $\Psi_{k-1}(\Phi_{k-1}(z)) = f(z) = \Psi_k(\Phi_k(z))$, therefore $|\Psi_{k-1}(\Phi_{k-1}(z))| = |\Psi_k(\Phi_k(z))|$, and since each of Ψ_{k-1} and Ψ_k is a piecewise winding map, it follows that $|\Phi_{k-1}(z)| = |\Phi_k(z)|$, and consequently, $\Phi_{k-1}(z) = \Phi_k(z)$. Points $z \in W_{2n} \bigcap W_1$ are treated in the same way.

Let $W = \bigcup_{k=1}^{2n} W_k$. Since Φ_{k-1} and Φ_k agree on $W_{k-1} \cap W_k$, and since each Φ_k is a homeomorphism of W_k onto W'_k, it follows that $\Phi(z) = \Phi_k(z)$ if $z \in W_k, k = 1,2,\ldots,2n$, defines a map Φ which maps W homeomorphically onto $D_r = \bigcup_{k=1}^{2n} W'_k$. Similarly, $\Psi(z) = \Psi_k(z)$ if $z \in W'_k$ defines a piecewise winding map of D_r onto itself. Finally, $f|W_k = \Psi_k \circ \Phi_k, k = 1,2,\ldots,2n$, implies that $f|W = \Psi \circ \Phi$, and consequently, $f|W$ is a simple umbrella map. □

10.2.1 BF-Maps Which Are Locally Different from the Canonical Maps

We describe now BF-maps $f : \mathbb{C} \to \mathbb{C}$ with critical sets in the real axes and which are not topologically equivalent to any of the canonical maps, which were described in 10.1.6–10.1.9, in any neighborhood of the point $z = 0$.

Each of the sets $E_n = \{(1/n, y) : 0 \le y \le 1/n\}$, $n = 1,2,\ldots$, is a line segment in $\mathbb{C}$, which is orthogonal to the real axis R and which has an end point at R. Let h be a conformal map of the upper half plane U onto the domain $D = U \setminus \bigcup_{n=1}^{\infty} E_n$ such that $h(z) \to 0$ as $z \to 0$ in U. Then h has a continuous extension to $\bar{U}$, which is denoted again by h.

Every point on ∂D has either one preimage point or two. Hence, h is discrete in $\bar{U}$ and homeomorphic in U. Let g be the standard folding map $(x,y) \to (x,|y|)$ and $f = h \circ g$. Then f is continuous and discrete in $\bar{U}$, open and sense preserving in U, and open and sense preserving in the lower half plane L. It is clear that f is not topologically equivalent to an umbrella map at any neighborhood of the point $z = 0$. Other canonical maps need not be mentioned since each of them is a special umbrella map.

10.2.2 Discrete Maps Which Are Not BF-Maps

There are many ways to construct discrete maps in $\mathbb{R}^2$ which are not BF-maps. We will present here only a few examples, each of which has a critical set which is different from the critical set of a BF-map in a different way.

Example (a). Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $\phi(x) = 2x + x\sin x$ and $f(x,y) = (\phi(x), y)$. Then $f : \mathbb{C} \to \mathbb{C}$ is a discrete map whose critical set is a union of lines which are parallel to the y-axis and cluster at the y-axis; hence, it is not a BF-map.

One can use this idea to construct a discrete map $f : \mathbb{R}^2 \to \mathbb{R}^2$, whose critical set is a union of a Cantor type set of lines parallel to the y-axis.

Example (b). Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be as in Example (a), and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(re^{it}) = re^{i\phi(t)}$ for $|t| \leq \pi$. Then f is continuous and discrete in $\mathbb{R}^2$, and the critical set of f is a union of infinitely many rays emanating from the point $z = 0$ and clustering on the positive half of the real axis.

One can use this idea to construct a discrete map $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose cluster set is a union of rays in a Cantor type set of directions.

Example (c). Let D be a triangle in $\mathbb{C}$ with vertices at the points $-1, 1$ and i, and D' the triangle with vertices $-1, 1$ and 2i. Let $f : \mathbb{C} \to \mathbb{C}$ be the piecewise linear map for which $f(z) = z$ if $z \in \mathbb{C} \setminus D'$ and $f(z) = \bar{z}$ if $z \in D$.

Then f is a BF-map with critical set $\Sigma_f = \partial D$, which is sense reversing in D and sense preserving in $\mathbb{C} \setminus D'$. The points -1 and 1 are $(3,1)$-cusp points, and every point on ∂D is a folding point.

Next, let ϕ be an automorphism of $\mathbb{C}$. Then $\phi \circ f \circ \phi^{-1}$ is a BF-map whose critical set is $\phi(\partial D)$, and its restriction to $\phi(\mathbb{C} \setminus D')$ is the identity map. Finally, let $\{\phi_\alpha : \alpha \in A\}$ be a family of automorphisms of $\mathbb{C}$ such that the sets $\phi_\alpha(D')$ are disjoint, and let $F : \mathbb{C} \to \mathbb{C}$ be defined by $F(z) = \phi_\alpha \circ f \circ \phi_\alpha^{-1}(z)$ if $z \in \phi_\alpha(D')$, and $F(z) = z$ otherwise. Then F is a continuous and discrete map of $\mathbb{C}$ onto itself whose critical set is $\Sigma_F = \bigcup_{\alpha \in A} \phi_\alpha(\partial D)$. By choosing the automorphisms properly, the sets $\phi_\alpha(\partial D)$ may cluster in very strange ways yielding discrete maps which are not BF-maps.

10.2.3 The Degree of a BF-Map and Prime BF-Maps

The *degree of a continuous map* $f : S^2 \to S^2$ will be denoted by $\deg f$. If $f : D \to \mathbb{R}^2$ is a proper umbrella map, then f is topologically equivalent to a straight umbrella map $F : \mathbb{C} \to \mathbb{C}$, and F has a continuous extension $\tilde{F} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is homeomorphic to S^2. The degree of f is then defined by $\deg f = \deg \tilde{F}$.

Let G and D be domains in S^2 with $\bar{D} \subset G$, $f : G \to S^2$ a map, and y a point in $S^2 \setminus f(\partial D)$. The *local (topological) index* or the *local degree of f at y relative to D* is denoted by $\mu(y, f, D)$. Then $\mu(y, f, D)$ is a constant integer in every connected component of $S^2 \setminus f(\partial D)$; see [203], p. 16 or [191]. The *local topological index of f at point* $x \in G$ will be denoted by $i(x, f)$. For discrete maps $f : G \to S^2, G \subset S^2$, $i(x, f)$ can be defined as $i(x, f) = \mu(x, f, D)$, where D is any neighborhood of x with $\bar{D} \subset G$ and with $f^{-1}(f(x)) \cap D = \{x\}$.

The existence of such a neighborhood D follows from the continuity and the discreteness of f. Obviously, the notion of $i(x, f)$ applies also to maps $f : X \to Y$, where X and Y are surfaces. At a noncritical point x, $i(x,f) = 1$ or -1, depending on the orientation of f at x; $i(x,f) = 0$ at a folding point x, and $i(x,f) = (p-q)/2$ at a (p,q)-cusp point x. It can be easily verified that if $f : D \to Y$ is a proper umbrella map with a cusp at x, then $\deg f = i(x,f)$.

We say that a proper umbrella map f is *prime* if f cannot be written as a composition of a proper umbrella mapping g with a noninjective analytic function h.

By degree considerations, it is clear that if f is a proper umbrella map with $|\deg f| = 1$, then f is prime. The following theorem describes the case where $|\deg f| > 1$:

Theorem 10.3. *Let $f : D \to \mathbb{C}$ be a proper umbrella map with a vertex at the point a. If $|\deg f| = d > 1$, then there exists a proper prime umbrella map g with $g(a) = 0$ such that $f(z) = (g)^d(z) + f(a)$, $z \in D$.*

Proof. We will show that $(f(z) - f(a))^{1/d}$ has a single-valued branch $g(z)$, that $g(z)$ is a proper umbrella, and that $|\deg g| = |(\deg f)/d| = 1$. It will then follow that g is prime and that $f(z) = (g)^d(z) + f(a)$, as needed.

Since f is a proper umbrella map, there are homeomorphisms $\phi : D \to \mathbb{C}$ and $\psi : \mathbb{C} \to f(D)$ and a straight umbrella map $f_0 : \mathbb{C} \to \mathbb{C}$ such that $f = \psi \circ f_0 \circ \phi$. We may assume that the ray E_0 from $z = 0$ to ∞ along the positive half of the real axis consists of noncritical points for f_0 except for the initial point $z = 0$. Let $E = \phi^{-1}(E_0)$ and $E' = f(E)$. Then $E' \backslash \{f(a)\}$ has a neighborhood V where $w^{1/n}$ has a branch $g_0(w)$. Let $g(z) = g_0(f(z))$ for $z \in E$.

Fix $r > 0$. Then $\Phi(t) = f(\phi^{-1}(r\mathrm{e}^{\mathrm{i}t}))$, $0 \le t \le 2\pi$ defines a map F of the circle $S^1(r)$ of radius r onto a Jordan curve with $|\deg F| = |\deg f| = d$. Therefore,

$$\Psi(t) = \Phi^{1/d}(t) = |\Phi(t)|^{1/d}\mathrm{e}^{\mathrm{i}(\arg\Phi(t)/d}$$

has a continuation from the point $g(r)$ along $[0, 2\pi]$, and since $|\deg F| = d$, $\Psi(0) = \Psi(2\pi)$. Hence, the relation

$$g(\phi^{-1}(r\mathrm{e}^{\mathrm{i}t})) = \Psi(t), \quad 0 \le t \le 2\pi$$

defines a single-valued function g from $\phi^{-1}(S^1(r))$ to $\mathbb{C}$, such that $(g(z))^d = f(z) - f(a)$.

By doing the same for all $r > 0$ and letting $g(0) = 0$, we get a function $g : D \to \mathbb{C}$ such that $(g(z))^d = f(z) - f(a)$ for all $z \in D$. By the construction, g is a proper umbrella and $\deg g = (\deg f)/d$. Hence, $|\deg g| = 1$, and thus, g is a proper prime umbrella. □

Corollary 10.2. *A proper umbrella map f of* $\deg f \neq 0$ *is prime if and only if* $|\deg f| = 1$.

Theorem 10.4. *Let $f : D \to \mathbb{C}$ be a proper (p,q)-cusp map with a vertex at the point a and with $|\deg f| = d$.*

If $d = 0$, then there exists a proper folding map g with $g(a) = 0$ such that $f(z) = (g)^p(z) + f(a)$, $z \in D$, for some integer $p \ge 1$.

If $d > 1$, *then there exists a proper prime simple cusp map* g *with* $g(a) = 0$ *such that* $f(z) = (g)^d(z) + f(a)$, $z \in D$.

Proof. If $d = 0$, then f is topologically equivalent to a straight simple (p,p)-cusp map $f_{p,p}$ and, clearly, $f_{p,p} = f^p_{1,1}$, which proves the theorem for $d = 0$.

Suppose now that $d > 1$. By Theorem 10.3, there exists a prime umbrella map g with $g(a) = 0$ such that $f(z) = g^d(z) + f(a)$. It remains to show that g is a cusp map. Let E denote the critical set of f. Then E, $f(E)$, and consequently $g(E)$ are arcs, and thus, by Corollary 10.1, a is a cusp point for g. Clearly, all other points of E are folding points for g. Consequently, g is a cusp map, which is prime. This completes the proof. □

Corollary 10.3. *A proper* (p,q)*-cusp map* f *with* $p \neq q$ *is prime if and only if* $|\deg f| = 1$.

10.3 Alternating Beltrami Equations and FQR-Maps

Let D, E, and μ be as in Sect. 10.1.1. A BF-map $f : D \to \mathbb{C}$ will be called a *folded quasiregular* map (abbreviated as FQR-map) if f is locally quasiregular or anti-quasiregular at every noncritical point and at every isolated critical point. It follows from the definition of FQR-maps that every FQR-map satisfies an alternating Beltrami equation (B) with μ which is locally bounded in the white regions and such that $1/\mu$ is locally bounded in the black regions. And conversely, a solution of an alternating Beltrami equation with a μ satisfying these assumption is an FQR-map if it is discrete.

One property is clearly shared by classical Beltrami equations and alternating ones, namely, if f is a nonconstant solution of (B) in a domain D and if h is analytic in $f(D)$, then $h \circ f$ is a solution too.

In the classical case, every homeomorphic solution is unique up to a conformal mapping and generates the set of all elementary solutions and thus may be considered as a prime solution. It is not known, however, whether every alternating Beltrami equation has what are called *prime solutions*, i.e., solutions f which cannot be written as a composition $h \circ g$ where h is a noninjective analytic function, and whether the set of all BF solutions is generated by prime solutions.

10.3.1 Proper Folding, Cusp, and Umbrella Solutions

A function $f : D \to \mathbb{C}$ is called a *proper folding solution*, resp., a *proper* (p,q)*-cusp* or a *proper umbrella*, if f is a solution to (B) in D and f is a proper folding mapping, resp., a proper (p,q)-cusp or a proper umbrella map. One of the main results in [233] asserts that every solution which belongs to one of the following families of solutions is prime and generates the set of all BF solutions:

(a) Proper folding solutions
(b) Proper (p,q)-cusp solutions with $|p-q| = 2$
(c) Proper umbrella solutions of degree 1 or -1

These results may be viewed as uniqueness and representation theorems and are the counterparts of the classical theory (see Chap. 3) and imply a strong rigidity property, namely, if f is a solution of a Beltrami equation (B) of any type mentioned above in (a)–(c), then every other solution of (B) with the same μ preserves the equivalence relation $z_1 \sim z_2$ if $f(z_1) = f(z_2)$, which is determined by f. This phenomenon does not occur in the classical case where the prime solutions are homeomorphisms which determine only the trivial equivalence. The rigidity property implies that an alternating Beltrami equation cannot have solutions of different types, as, for instance, prime (p,q)-cusp solutions with different pairs (p,q). Furthermore, one can show that if f is a proper (p,q)-cusp solution with $|p-q| = 2d > 2$ or a proper umbrella solution with $|\deg f| = d > 1$, then B has a prime solution g which is a proper (r,s)-cusp solution with $|r-s| = 2$, or, respectively, a prime umbrella solution of degree 1 or -1, such that $f(z) = (g(z))^d$.

Theorem 10.5. *Let D be a simply connected domain in $\mathbb{C}$, E a crosscut in D with $m_2(E) = 0$, D_1 and D_2 the two connected components of $D \backslash E$ and $\mu : D \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ a measurable function such μ is locally bounded in D_1 and $1/\mu$ is locally bounded in D_2, and let $w = f(z)$ be a proper folding solution of the Beltrami equation (B).*

If $w = g(z)$ is a solution of (B) in D, then there exists a function h which is continuous in $f(D)$ and analytic in $f(D_1) = f(D_2)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.

Proof. Let $f_i = f|D_i$ and $h_i = g \circ f_i^{-1}$, $i = 1,2$. Since μ is locally bounded D_1 and $1/\mu$ is locally bounded in D_2, it follows that f and g are sense preserving in D_1 and sense-reversing in D_2. Therefore, both h_1 and h_2 are sense preserving. Since μ is locally bounded in D_1 and $1/\mu$ is locally bounded in D_2, it follows that f_1 and f_2 are locally quasiconformal and therefore have generalized derivatives which are locally square integrable, and so does g, being a solution of (B). Hence, h_1 and h_2 have generalized derivatives which are locally integrable. Since f and g have a.e. the same complex dilatation μ, we get by a standard computation of the dilatation of h_i that h_i satisfies a.e. the Cauchy–Riemann equations, $i = 1,2$. In view of the local integrability of the derivatives of h_i, we obtain by Weyl's lemma that h_i is analytic in $f(D_i), i = 1,2$.

Since f is a folding, f_1 and f_2 have, respectively, homeomorphic extensions to $D_1 \cup E$ and to $D_2 \cup E$. It thus follows that h_1 and h_2 have the same boundary values on the set $f(E)$ and that $f(E)$ is a free arc in the boundary of $f(D_1)$. Hence, h_1 and h_2 represent the same analytic function, denoted by h, in $f(D_1)$. The last implication follows, for instance, by an application of the two constant theorem. Clearly, h has a continuous extension to $f(D_1) \cup f(E)$, which is denoted again by h, and $g = h \circ f$.

The statement that $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$ is a trivial consequence of the fact that $g = h \circ f$. This completes the proof. □

Remark 10.1. As a consequence of Theorem 10.5, one obtains the following result: If (B) has a folding solution f, then any other solution of (B) will preserve the equivalence relation $z_1 \sim z_2$ if $f(z_1) = f(z_2)$.

Theorem 10.6. *Let D, E, D_1, D_2, and μ be as in Theorem 10.5, and let $w = f(z)$ be a proper (3,1)-cusp solution of the Beltrami equation (B) in D. If $w = g(z)$ is a solution of (B) in D, then there exists a function h which is analytic in $f(D)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.*

Proof. Since f is a proper (3,1)-cusp map in D, there exist homeomorphisms ϕ of D onto $\mathbb{C}$ and ψ of $\mathbb{C}$ onto $f(D)$ such that $f = \psi \circ f_{3,1} \circ \phi$, $\phi(E) = R$ and $\psi(R) = f(E)$. Here, $f_{3,1}$ is the straight $(3,1)$-cusp map; see 10.1.8. Let $z_0 = \phi^{-1}(0), w_0 = f(z_0), D' = f(D)$ and $E' = f(E)$. Then $D' \backslash E'$ has two components D'_1 and D'_2, and $E' \backslash w_0$ has two components E'_1 and E'_2. For $k = 1,2,3$, let

$$S_k = \{z \in \mathbb{C} : (k-1)\pi/3 < \arg\, z < k\pi/3\},$$

and $G_k = \phi^{-1}(S_k)$, and let $G_4 = D_2$.

Then $f(G_1) = f(G_3) = f(G_4) = D'_1$, and $f(G_2) = D_2$. Now, let $f_j = f|G_j$ and $h_j = g \circ f_j, j = 1,2,3,4$. Since f and g are solutions of (B), then as shown in the proof of Theorem 10.5, all h_j are analytic. The functions h_1 and h_4 are defined in D'_1 and have the same boundary values on one component, say, E'_1 of $E' \backslash w_0$ and h_3 and h_4 have the same boundary values on E'_2. Hence, $h_1 = h_4 = h_3$ in D'_1.

We show now that h_2 is an analytic continuation of h_1 across E'_2 and of h_3 across E'_1. Indeed, let a' be a point on E'_2. Let a be the unique point in D_1 such that $f(a) = a'$. Since f is a local homeomorphism at a, a has a neighborhood W which is mapped homeomorphically by f onto a neighborhood V of a'. Then, as above, $h_V = g \circ (f|W)^{-1}$ is analytic in V, and $h_V = h_j$ in $V \cap D'_j, j = 1,2$. It thus follows that h_2 is an analytic continuation of h_1 across E'_2. Similarly, h_2 is an analytic continuation of h_3 across E'_1. Then $h_j, j = 1,\ldots,4$, define a function h which is analytic in $f(D) \backslash w_0$ such that $h|D'_1 = h_1 = h_3 = h_4$ and $h|D'_2 = h_2$. Obviously, w_0 is removable, and hence, h has a continuous extension to D', which is denoted again by h, and h is analytic in w_0 too, $g = h \circ f$ in D, and $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$. □

Corollary 10.4. *Let D, E, D_1, D_2, and μ be as in Theorem 10.5. Then (B) cannot have both folding solutions and $(3,1)$-cusp solutions.*

Remark 10.2. By Theorems 10.5 and 10.6, it follows that if a Beltrami equation (B) has a solution f which is either a homeomorphism or a proper folding or a proper (3.1)-cusp, then f generates the BF solutions. However, not every (p,q)-cusp solution of (B) has this property, as can be seen in the following example. Below, we will characterize all (p,q)-cusp solutions and all umbrella solutions which have the generating property.

10.3.2 Example

We present here a function $\mu(z)$ for which equation (B) has a proper $(5,1)$-cusp solution f and a proper $(3,1)$-cusp solution g. Then, clearly, $g \neq h \circ f$ for any analytic function h, since deg $f = 2$ and deg $g = 1$. This shows that f does not generate the set of all BF solutions.

Let Φ and Ψ be automorphisms of $\mathbb{R}^2$ which are given in polar coordinates by $\Phi(r,\theta) = (r,\phi(\theta))$ and $\Psi(r,\theta) = (r,\psi(\theta))$, where ϕ and ψ are piecewise linear increasing self-maps of $[0,2\pi]$ such that $\phi([0,\pi/5]) = [0,\pi/3]$, $\phi([\pi/5,4\pi/5]) = [\pi/3,2\pi/3]$, $\phi([\pi/5,\pi]) = [2\pi/3,\pi]$, $\phi([\pi,2\pi]) = [\pi,2\pi]$, and $\psi([0,\pi]) = [0,\pi/2]$, $\psi([\pi,4\pi/3]) = [\pi/2,\pi]$, $\psi([4\pi/3,5\pi/3]) = [\pi,3\pi/2]$, $\psi([5\pi/3,2\pi]) = [3\pi/2,2\pi]$. Recall that $f_{3,1}(z) = z^3/|z|^2$ if Im $z > 0$ and $f_{3,1}(z) = \bar{z}$ if Im $z \leq 0$. Then $g = \Psi \circ f_{3,1} \circ \Phi$ is a proper $(3,1)$-cusp solution of (B) with $\mu = g_{\bar{z}}/g_z$. Next, let $f(z) = (g(z))^2$, $z \in \mathbb{C}$. Then f is a solution of (B) with the same μ, and f is topologically equivalent to the straight $(5,1)$-cusp map $f_{5,1}$. Consequently, f is a proper $(5,1)$-cusp solution.

Theorem 10.7. *Let D, E, D_1, D_2, and μ be as in Theorem 10.5 and let $w = f(z)$ be a proper $(p+2,p)$-cusp solution of the Beltrami equation (B) in D. If $w = g(z)$ is a BF solution of (B) in D, then there exists a function h which is analytic in $f(D)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.*

Proof. Let $z_0 \in E$ be the cusp point of f, and let E_1 and E_2 be the two components of $E \setminus \{z_0\}$. Denote $w_0 = f(z_0)$, $D' = f(D)$ and $E' = f(E)$.

We consider the two cases:

(i) p is odd.
(ii) p is even.

In the first case, we suppose that $p = 2n-1$, then $p+2 = 2n+1$. Since p is odd, f maps E homeomorphically onto E'. Let $E'_i = f(E_i), i = 1,2$.

As in the proof of Theorem 10.6, one sees that the set $f^{-1}(E')$ divides D_1 into $2n+1$ sectors $G_1,\ldots,G_{2n+1}$ and divides D_2 into $2n-1$ sectors $\tilde{G}_1,\ldots,\tilde{G}_{2n-1}$. We may assume that G_1 is adjacent to $\tilde{G}_1$ along E_1, and G_{2n+1} to $\tilde{G}_{2n-1}$ along E_2.

Let $f_i = f|G_i, h_i = g \circ f_i^{-1}, i = 1,\ldots,2n+1$, and $\tilde{f}_j = f|\tilde{G}_j$ $\tilde{h}_j = g \circ f_j^{-1}$, $j = 1,\ldots,2n-1$. Since f and g satisfy the same Beltrami equation, it follows as in Theorem 10.5 that all h_i and $\tilde{h}_j$ are analytic.

Since h_1 and $\tilde{h}_1$ have the same boundary value along $f(E_1)$, it follows that $h_1 = \tilde{h}_1$. Similarly, $h_{2n+1} = \tilde{h}_{2n-1}$. Now, h_2 is an analytic continuation of h_1 and $\tilde{h}_2$ of $\tilde{h}_1$. Since $h_1 = \tilde{h}_1$ and since h_2 and $\tilde{h}_2$ are defined in the same domain, it follows that $h_2 = \tilde{h}_2$. Similarly, $h_i = \tilde{h}_i$, $i = 3,\ldots,2n-1$. Starting now with $h_{2n+1} = \tilde{h}_{2n-1}$, the same argument yields $h_{2n} = \tilde{h}_{2n-2}$ and by continuing in the same way also $h_i = \tilde{h}_{i-2}$, $i = 3,\ldots,2n-1$. But $h_1 = \tilde{h}_1$. Therefore, $h_1 = h_3 = h_5 = \cdots$, and $h_2 = h_4 = h_6 = \cdots$, and similar relations hold for $\tilde{h}_1, \tilde{h}_2, \ldots$. As in Theorem 10.5, these relations determine a function h which is analytic in D', including the points of E', which coincides with h_1 in $f(D_1)$ and with h_2 in $f(D_2)$. Clearly, $g = h \circ f$.

In the second case, we suppose that $p = 2n$. Then $f(E_1) = f(E_2) = E'$ and $D'_0 = D' \backslash E'$ is a simply connected domain. The set $f^{-1}(D'_0)$ has $n+1$ connected components $G_1, \dots, G_{n+1}$ in D_1 and n components divide $\tilde{G}_1, \dots, \tilde{G}_n$ in D_2. We may assume that G_1 is adjacent to $\tilde{G}_1$ along E_1 and G_{n+1} to $\tilde{G}_n$ along E_2.

Let $f_i = f|G_i$, $h_i = g \circ f_i^{-1}$, $i = 1, \dots, n+1$, and $\tilde{f}_j = f|\tilde{G}_j$, $\tilde{h}_j = g \circ f_j^{-1}$, $j = 1, \dots, n$. Since f and g satisfy the same Beltrami equation, it follows as above that all h_i and $\tilde{h}_j$ are analytic. Since h_1 and $\tilde{h}_1$ have the same boundary value along $f(E_1)$, it follows that $h_1 = \tilde{h}_1$. Similarly, $h_{n+1} = \tilde{h}_n$. Now, h_2 is an analytic continuation of h_1 and $\tilde{h}_2$ of $\tilde{h}_1$. Since $h_1 = \tilde{h}_1$, and since h_2 and $\tilde{h}_2$ are defined in the same domain, it follows that $h_2 = \tilde{h}_2$. Similarly, $h_i = \tilde{h}_i$, $i = 3, \dots, n$. Starting with $h_{n+1} = \tilde{h}_n$, the same argument yields $h_n = \tilde{h}_{n-1}, h_{n-1} = \tilde{h}_{n-2}$ and $h_i = \tilde{h}_{i-1}$, $i = 3, \dots, n$. But $h_1 = \tilde{h}_1$; hence, $h_{n+1} = \cdots = h_2 = h_1 = \tilde{h}_1 = \tilde{h}_2 = \cdots = \tilde{h}_n$. These relations define a function h which is analytic in $f(D)$, including the points of E', which satisfies $g = h \circ f$. The analyticity of h at a point $w \in E' \backslash \{w_0\}$ follows from the fact that $f^{-1}(w)$ meets D_1, and w_0 is an isolated removable singularity. In both cases, $g = h \circ f$ implies $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$. □

Corollary 10.5. *Let f be a proper (p,q)-cusp solution of (B) in a domain D. If $|p-q| = 2$, then f generates the solution set.*

Theorem 10.8. *Let $w = f(z)$ be a proper (p,q)-cusp solution of (B) in a domain D. If $|p-q| = 2$, then f is prime.*

Proof. A simple computation shows that $|p-q| = 2$ implies $|\deg f| = 1$, which, in turn, shows that f is prime. □

The following theorem shows that a proper (p,q)-cusp solution with $|p-q| > 2$ is not prime and can be factored as $h \circ g$, where g is a proper prime cusp solution and h is an analytic power map, and in particular f is not prime:

Theorem 10.9. *Let f be a proper (p,q)-cusp solution of (B) in a domain D with $|p-q| = 2d > 2$ such that its cusp point is at $z = 0$ and $f(0) = 0$. If $d > 1$, then there exists a proper prime (n,m)-cusp solution g of (B) in D such that $f(z) = (g(z))^d$, $z \in D$.*

Proof. Since $|\deg\ f| = |p-q|/2 = d$, it follows by Theorem 2.12 that $f(z) = (g(z))^d$, where g is a proper (m,n)-cusp map with $|m-n| = 2$. The power map $z \to z^d$ is analytic; therefore, g is a solution of (B) too. The relation $|m-n| = 2$ implies $|\deg\ g| = 1$, and consequently, g is a proper prime cusp solution. □

We now turn to proper umbrella solutions of degree 1 or -1. It is shown in [233] that such a solution is prime and generates the set of all BF solutions.

Lemma 10.4. *If $f : S^1 \to S^1$ is a piecewise injective map of nonzero degree, then every fiber $f^{-1}(w), w \in S^1$, contains at least one noncritical point.*

Proof. For $z \in S^1$, let $i(z,f)$ denote the local topological index of f at z; see Sect. 10.2.3. Then $i(z,f) = 0$ if z is a critical point, and $i(z,f) = 1$ or -1 otherwise. Fix $w \in S^1$. The set S^1 is compact; hence, f is proper and

$$\deg f = \sum_{z \in f^{-1}(w)} i(z,f).$$

Since $\deg f \neq 0$, it follows that $i(z,f) \neq 0$ at least for one point z in $f^{-1}(w)$ and so z is noncritical. □

Lemma 10.5. *Let J be a Jordan curve in $\mathbb{C}$, $F : S^1 \to J$ a piecewise injective map with critical points $s_k = \mathrm{e}^{\mathrm{i}t_k}, k = 1,2,\ldots,2n$,*

$$0 < t_1 < t_2 < \cdots < t_{2n} < 2\pi.$$

Let $\gamma : [0,2\pi] \to J$ be defined by $\gamma(t) = F(\mathrm{e}^{\mathrm{i}t}), 0 \le t \le 2\pi$.

For each $t \in [0,2\pi]$, let (h_t, D_t) be an analytic function element at the point $\gamma(t)$, where $D_t = \{z : |z-\gamma(t)| \le r_t\}$ for some $r_t > 0$, $0 \le t \le 2\pi$, such that $\{h_t : 0 \le t \le 2\pi\}$ defines an analytic continuation from (h_0, D_0) to $(h_{2\pi}, D_{2\pi})$ along the path γ.

If $(h_0, D_0) = (h_{2\pi}, D_{2\pi})$ and if $|\deg f| = 1$, then there exist a neighborhood V of J and a (single-valued) analytic function h in V such that $h = h_t$ in $V \cap D_t$ for all $t \in [0,2\pi]$.

Proof. By induction on n.

If $n = 0$, then F has no critical points, and since $|\deg f| = 1$, it follows that F is a homeomorphism of S^1 onto J.

Since J is compact and can be mapped tamely onto S^1, we may replace the function elements (h_t, D_t), $0 \le t \le 2\pi$ by a finite chain of function element (h_j, G_j), $j = 1,2,\ldots,m$, such that each is a disc centered at a point on J and contained in one of the discs D_t, $h_j = h_t|G_j$, $G_1 \subset D_0$ and $G_m \subset D_{2\pi}$, G_j meets only G_{j-1} and G_{j+1} and such that h_j is a direct analytic continuation of h_{j-1}, $j = 1,2,\ldots,m$. Then $V = \bigcup_{j=1}^m G_j$ is a neighborhood of J. Since $(h_0, D_0) = (h_{2\pi}, D_{2\pi})$, it follows by the way that the chain (h_j, G_j), $j = 1,2,\ldots,m$ was constructed such that the function h which is defined by $h(z) = h_j(z)$ for $z \in G_j$ is a well-defined (single-valued) analytic function in V, and the lemma follows.

Suppose now that the lemma holds for functions from S^1 into J with $2n-2$ critical points, and let $F : S^1 \to J$ have $2n$ critical points. The reduction of F to a function with $2m-2$ critical points will be done by a certain surgery on a subarc of S^1 which is folded by F.

First, note that F has a critical point $\mathrm{e}^{\mathrm{i}t_k}$ such that F is sense-reversing at $\mathrm{e}^{\mathrm{i}(t_k - t)}$ and sense preserving at $\mathrm{e}^{\mathrm{i}(t_k + t)}$ for all sufficiently small $t > 0$.

Choose a sense-preserving parametric representation $\tilde{\gamma} : [0,2\pi] \to J$ of J with $\tilde{\gamma}(0) = F(\mathrm{e}^{\mathrm{i}t_k})$. Let t_0 be the maximal of numbers for which the restriction $\tilde{\gamma}|[0,t_0]$ has two lifts γ^- and γ^+ from the point $\mathrm{e}^{\mathrm{i}tk}$. Let $\gamma^-(t_0) = \mathrm{e}^{\mathrm{i}\alpha}$ and $\gamma^+(t_0) = \mathrm{e}^{\mathrm{i}\beta}$.

Then $\alpha < t < \beta$ and $\beta - \alpha < 2\pi$. Note that $e^{i\alpha} \neq e^{i\beta}$, since otherwise $\deg F = 0$, contradicting the assumption that $|\deg F| = 1$. By the maximality of t_0, it follows that $e^{i\alpha}$ or $e^{i\beta}$ is a critical point.

In view of the assumption that $(h_0, D_0) = (h_{2\pi}, D_{2\pi})$, we may replace F by $\phi \circ F$ for any sense-preserving automorphism ϕ of S^1. Hence, we may assume that $t_k = \pi, \alpha = \pi/2, \beta = 3\pi/2$ and that $F(e^{-it}) = F(e^{it})$ for all $t \in [\pi/2, \pi]$. Then by uniqueness of analytic continuation, $h_{\pi-t} = h_{\pi+t}$ in $D_{\pi-t} \cup D_{\pi+t}$ for all $t \in [0, \pi/2]$, and in particular, $h_{\pi/2} = h_{3\pi/2}$ in a neighborhood of the point $F(i) = F(-i)$.

We now replace F by a function $\tilde{F} : S^1 \to J$ which has $2n-2$ critical points, and the family (h_t, D_t) by a family $(\tilde{h}_t, \tilde{D}_t)$, as follows.

Define $\tilde{F} : S^1 \to J$ by letting $\tilde{F}(e^{it}) = F(e^{it/2})$ if $|t| < \pi/2$ and $\tilde{F}(-i) = F(i)$.

Now, F is continuous in $\{e^{it} : |t| \leq \pi/2\}$ and $F(i) = F(-i)$; therefore, $\tilde{F}$ is continuous. Recalling that i or $-i$ is a critical point of F, therefore, $\tilde{F}$ has $2n-2$ critical points and $\deg \tilde{F} = \deg F = 1$ or -1.

Now, define $(\tilde{h}_t, \tilde{D}_t) = (h_{t/2}, D_{t/2})$ for $t \in [0, \pi]$ and $(\tilde{h}_t, \tilde{D}_t) = (h_{(2\pi+t)/2}, D_{(2\pi+t)/2})$ for $t \in [\pi, 2\pi]$.

Note that $(h_{\pi/2}, D_{\pi/2}) = (h_{3\pi/2}, D_{3\pi/2})$, since they are obtained by continuation of (h_π, D_π) along the same arc. Therefore, $(\tilde{h}_{2\pi}, \tilde{D}_{2\pi})$ is well defined, and consequently, $(\tilde{h}_t, \tilde{D}_t)$ and $\tilde{F}$ satisfy the assumptions of the lemma.

By the induction hypothesis, there exist a neighborhood V of J and a single-valued analytic function h in V such that $h(z) = \tilde{h}_t(z)$ for all $z \in V \cap \tilde{D}_t$, $t \in [0, 2\pi]$.

Now, for $t \in [-\pi/2, \pi/2], (h_t, D_t) = (\tilde{h}_{t/2}, \tilde{D}_{t/2})$, and hence, $h(z) = h_t(z)$ for all $z \in V \cap D_t, |t| \leq \pi/2$. It remains to show that $h(z) = h_t(z)$ for $\pi/2 \leq t \leq 3\pi/2$. This follows from the fact that $(h_t, D_t), \pi/2 \leq t \leq \pi$ and $(h_t, D_t), \pi \leq t \leq 3\pi/2$ are analytic continuations along the same curve, since $F(e^{i(\pi-t)}) = F(e^{i(\pi+t)})$ for $t \in [0, \pi/2]$ and the fact that the two curves mentioned above, i.e., $\{\gamma(t) : \pi/2 \leq t \leq \pi\}$ and $\{\gamma(t) : \pi \leq t \leq 3\pi/2\}$, are contained in V and the function elements $(h_{\pi/2}, D_{\pi/2}) = (h_{3\pi/2}, D_{3\pi/2})$ coincide with the value of h in a neighborhood of the point $F(i) = F(-i)$. Consequently, $h(z) = h_t(z)$ for all $z \in V \cap D_t$ for $\pi/2 \leq t \leq 3\pi/2$, too. This completes the proof. □

10.3.3 Definition

A piecewise analytic continuation along a path $\gamma : [a,b] \to \mathbb{C}$ with degeneration at the points t_i, $i = 1, 2, \ldots, n$, $0 < t_1 < t_2 < \cdots < t_n < b$ is a one-parameter family of maps $f_t : D_t \to \mathbb{C}$, $a \leq t \leq b$ such that:

(a) $D_t = \{z : |z - \gamma(t)| \leq r(t)\}$, where $r(t) = 0$ if $t \in T = \{t_1, t_2, \ldots, t_n\}$ and $r(t) > 0$ if $t \in [a,b] \backslash T$.
(b) The function $t \to f_t(\gamma(t))$ is continuous in $[a,b]$.
(c) Each f_t is analytic in $\operatorname{Int} D_t$.
(d) If $t < t'$ in the same connected component of $[a,b] \backslash T$, then f_t is an analytic continuation of $f_{t'}$ along the sub-path $\gamma|[t,t']$.

Lemma 10.6. *Let J be a Jordan curve, $F: S^1 \to J$ a piecewise injective map with $|\deg F| = 1$ and with critical points $\mathrm{e}^{\mathrm{i}t_j}$, $j = 1, 2, \ldots, 2n$, where $0 < t_1 < t_2 < \cdots < t_{2n} < 2\pi$.*

Let $h_t, 0 \leq t \leq 2\pi$ be a piecewise analytic continuation along the path $\gamma : [0, 2\pi] \to J$, which is defined by $\gamma(t) = F(\mathrm{e}^{\mathrm{i}t})$, $0 \leq t \leq 2\pi$ with degeneration at the points $t_1, t_2, \ldots, t_{2n}$ and with $h_0 = h_{2\pi}$.

Suppose that for every point of degeneration t_j there is $\delta_j > 0$ such that $h_t(\gamma(t)) = h_{t'}(\gamma(t'))$ for every t and t', $t_j - \delta < t < t_j < t'_j < t_j + \delta$ with $\gamma(t) = \gamma(t')$.

Then there exist a neighborhood V of J and an analytic function h in V such that $h = h_t$ in $V \cap D_t$ for all $t \in [a, b]$.

Proof. Since $|\deg F| \neq 0$, it follows by Lemma 10.4 that for each critical point $\mathrm{e}^{\mathrm{i}t_k}$ of F, there is a noncritical point $\mathrm{e}^{\mathrm{i}t'_k}$ such that $F(\mathrm{e}^{\mathrm{i}t_k}) = F(\mathrm{e}^{\mathrm{i}t'_k})$, $k = 1, 2, \ldots, 2n$. For each k, choose real numbers t_k^+ and t_k^- such that $t_k - \delta_k < t_k^- < t_k < t_k^+ < t_k + \delta_k$, such that $\Gamma_k = \{F(t) : t_k^- \leq t \leq t_k^+\} \subset D_{t'_k}$, and such that the arcs Γ_k, $k = 1, 2, \ldots, 2n$ are disjoint. We now apply the same kind of surgery that we performed on the arc $\{\mathrm{e}^{\mathrm{i}t} : \pi/2 \leq t \leq 3\pi/2\}$ in the induction process in the proof of Lemma 10.5 to each of the arcs Γ_k, $k = 1, 2, \ldots, 2n$, and get a piecewise injective map $\tilde{F}: S^1 \to J$ and a one-parameter family of function elements $(\tilde{h}_t, \tilde{D}_t), 0 \leq t \leq 2\pi$ which satisfy the assumptions of Lemma 10.5.

By Lemma 10.5, there exist a neighborhood V of J and a single-valued analytic function h in V such that $h(z) = \tilde{h}_t(z)$ for all $z \in V \cap \tilde{D}_t$ for all $t \in [0, 2\pi]$. Since each $\Gamma_k \subset D_{t'_k} \cap J \subset V \cap D_{t'_k}$, $k = 1, 2, \ldots, 2n$, it follows that $h(z) = h_t(z)$ for all $z \in V \cap D_t$ for all $t \in [0, 2\pi]$. This completes the proof of the lemma. □

Theorem 10.10. *Let $f: D \to \mathbb{C}$ be a proper umbrella solution of the equation (B). If $|\deg f| = 1$, then f generates the solution set.*

Furthermore, if g is a solution, then $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.

Proof. Let g be a solution of (B). We will show that $g = h \circ f$ for some analytic function h. The case where f is a homeomorphism follows from the Theorem 3.1.

Suppose that f is not injective. Then $|\deg f| = 1$ implies that the critical set E of f is not a point, and hence, it is a star. With no loss of generality, we may assume that the vertex of E is at $z = 0$ and that $f(0) = 0$.

Since f is a proper umbrella map, there are homeomorphisms ϕ of $\mathbb{C}$ onto D and ψ onto $\mathbb{C}$ of $D' = f(D)$ such that $\phi(0) = 0$, $\psi(0) = 0$, $\phi^{-1}(E)$ is a straight star and $\tilde{f} = \psi \circ f \circ \phi$ is a straight umbrella map, which maps each circle $S^1(r) = \{z \in \mathbb{C} : |z| = r\}$ onto itself. Since $|\deg f| = 1$ and ϕ and ψ are homeomorphisms, it follows that

$$|\deg \tilde{f}| = |\deg f \circ \phi| = |\deg (f \circ \phi)|S^1(r)| = 1$$

for all $r > 0$. We may assume that $\phi^{-1}(E)$ does not contain the positive half of the real axis or, in other words, that every real and positive z is a noncritical point of f.

Fix $r > 0$. For each point $s \in S^1(r) \backslash \phi^{-1}(E)$, the point $\phi(s)$ has a neighborhood U_s which is mapped by f homeomorphically onto a disc V_s centered at the point $f(\phi(s))$.

Let $f_s = f|U_s$ and $h_t = g \circ (f|U_s)^{-1}$ for $t \in [0, 2\pi]$ with $s = re^{it}$. The functions g and f satisfy the same Beltrami equation in U_s and $||\mu|U_s||_\infty < 1$; therefore, h_t is analytic.

Every point s' in $S^1(r) \cap \phi^{-1}(E)$ is a folding point for $\tilde{f}$, and the point $\phi(s')$ is a folding point for f. By Lemma 10.2, the point $\phi(s')$ has a neighborhood $U_{s'}$, where f is a proper folding map, and by Theorem 10.5, there is a continuous function $h_{t'}$ in $V_{s'} = f(U_{s'})$ which is analytic in int$V_{s'}$, such that $g|U_{s'} = h_{t'} \circ (f|U_{s'})$ where $s' = re^{it'}$.

Let $F = f \circ \phi, D_t = \bar{V}_t$ for $t \in [0, 2\pi]$ with $\phi(re^{it}) \in D \backslash E$ and $D_t = \{f(\phi(re^{it}))\}$ for $t \in [0, 2\pi]$ with $\phi(re^{it}) \in D \cap E$. Then $J = F(S^1(r))$ is a Jordan curve, $F : S^1(r) \to J$ is a piecewise injective map with critical points at the points of $\phi^{-1}(E) \cap S^1(r)$, and $\{(h_t, D_t) : 0 \leq t \leq 2\pi\}$ is a piecewise injective analytic continuation with degenerations. Since $(h_0, D_0) = (h_{2\pi}, D_{2\pi})$ and $|\deg\, F| = 1$, it follows, by Lemma 10.6, that there exist a neighborhood W_r of J and a singe-valued analytic function H_r such that $H_r(z) = h_t(z)$ for all $z \in D_t$ for all $t \in [0, 2\pi]$. As a result of the relations $g = h_t \circ (f|U_s)$, where $s = re^{it}$, we have $g|W_r = H_r \circ (f|W_r)$. The last relation holds for all $r > 0$, and clearly, $H_{r'}$ is an analytic continuation of H_r for all $0 < r < r'$; therefore, the function $H_r, r > 0$ defines a singe-valued analytic function h in $f(D) \setminus \{0\}$ such that $g = h \circ f$ in $D \setminus \{0\}$. The point 0 is removable for h and therefore $g = h \circ f$ everywhere in D.

The relation $g = h \circ f$ implies that $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$. This completes the proof. □

Theorem 10.11. *Let $f : D \to \mathbb{C}$ be a proper umbrella solution of equation (B) with $|\deg\, f| = d \geq 1$ and with a vertex at the point z_0:*

(i) If $d = 1$, then f is a prime solution.
(ii) If $d > 1$, then (B) has a proper prime solution $w = g(z)$ in D such that $f(z) = (g(z))^d + f(z_0)$.

Proof. (i) follows from the fact that if $f = h \circ g$ where h is analytic, then both h and g are proper and $\deg\, f = (\deg\, h) \cdot (\deg\, g)$. Then $|\deg\, f| = 1$ implies that $|\deg\, h| = 1$ and hence h is injective. Consequently, f is prime.

(ii) By Theorem 10.6, there exists a proper prime umbrella map g such that $f(z) = (g(z))^d + f(z_0)$. The power map $z \to z^d$ is analytic; therefore, g is a solution too, and thus, g is a proper prime solution.

□

10.4 Existence of Local Folding Solutions

10.4.1 Assumptions on the Equation and Geometric Configuration

Let $\mu : D \to \bar{\mathbb{C}}$ be a measurable function in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that μ is locally bounded in $D_1 = \{z \in D : \text{Im}\, z > 0\}$ and such that $1/\mu$ is locally bounded in $D_2 = \{z \in D : \text{Im}\, z < 0\}$. Let $E = D \cap \mathbb{R}$.

Theorem 10.12. *(Existence of local folding solutions). Let D, D_1, D_2, E, and μ be as in Theorem 10.4.1. If there exist $r \in (0,1)$, a nonnegative integer m, and a complex-valued real analytic function $Q(x,y)$ in $|z| < r$, $z = x + \mathrm{i}y$, with*

(a) $$\operatorname{Re} Q(0,0) > 0$$

such that for $|z| < r$,

(b) $$\mu(z) = 1 - y^{2m+1} Q(x,y),$$

then (B) has a folding solution in some neighborhood of 0.

Proof. In [234], Theorem 1.1, it was considered a Beltrami equation (B) with

$$\mu(z) = \mathrm{e}^{2\mathrm{i}\theta(z)}[1 - \rho(y)M(z)], \tag{M}$$

where θ and ρ are real-valued functions and M is a complex-valued function which satisfy certain conditions. We then applied a process, which we called a deformation of the complex dilatation, and reduced (B) to a Beltrami equation in the (ξ, η) plane with complex dilatation

$$\tilde{\mu}(\xi, \eta) = 1 - \rho(\eta)\tilde{M}(\xi, \eta),$$

where ρ is the same as in (M) and $\tilde{M}$ satisfies a similar condition as M; see [234], pp. 475–477. The same method can be applied here. Thus, we will assume that $\theta(z) \equiv 0$. We will also assume that $m = 0$ and that M is a polynomial. The general case is proved in a similar way.

In view of (a), M has the form

$$M(x,y) = 2a + 2\mathrm{i}b + \sum_{n+k=1}^{N} q_{nk} z^n \bar{z}^k, \tag{10.4.1}$$

where $N \geq 1$, a and b are real, and $a > 0$.

As in [150, pp. 23–24], we set in (B)

$$w = f(z) = z + c_{01}\bar{z} + \sum_{n+k=2}^{\infty} c_{nk} z^n \bar{z}^k \tag{10.4.2}$$

with $c_{n0} = 0$ for all $n > 1$. On applying of (M), one obtains all c_{nk} (uniquely). Again, exactly as in [150], one can show that the series in (10.4.2) converges uniformly in $U_0 = \{z : |z| < r_0\}$ for some $r_0 \in (0, r)$.

It remains to show that f is a folding in some neighborhood U of 0 which is contained in U_0. Let $f = u + \mathrm{i}v$; then, by (10.4.2),

$$u = 2x - b(x^2 + y^2) + \sum_{n+k=3}^{\infty} \alpha_{nk} x^n y^k \tag{10.4.3}$$

and

$$v = a(x^2 + y^2) + \sum_{n+k=3}^{\infty} \beta_{nk} x^n y^k \tag{10.4.4}$$

for some α_{nk} and β_{nk} which depend on the c_{nk}'s.

Next, define $\varphi : U_0 \to \mathbb{C}$ by letting

$$\varphi(x,y) = (u(x,y),y) \tag{10.4.5}$$

with u as in (10.4.3). Then φ has a nonzero Jacobian at 0, and hence, it is a diffeomorphism in some neighborhood $U_1 \subset U_0$ of 0. Consequently, there exists a mapping $F : \varphi(U_1) \to \mathbb{C}$ such that $f = F \circ \varphi$. Since $\varphi|U_1$ is injective, it suffices to show that F is a folding. Now, $(\varphi|U_1)^{-1}$ has the form

$$\varphi^{-1}(u,y) = (x(u,y),y); \tag{10.4.6}$$

hence,

$$F(u,y) = (u, \tilde{v}(u,y)), \tag{10.4.7}$$

where, by (10.4.4),

$$\tilde{v}(u,y) = v(x(u,y),y) = a[x^2(u,y) + y^2] + \sum_{n+k=3}^{\infty} \beta_{nk} x^n(u,y) y^k, \tag{10.4.8}$$

or

$$\tilde{v}(u,y) = A_0(u) + A_1(u)y + A_2(u)y^2 + \sum_{k=3}^{\infty} A_k(u) y^k \tag{10.4.9}$$

for some functions $A_k(u)$.

We now show that $A_1(u) \equiv 0$ and $A_2(u) \neq 0$ in some neighborhood of 0. This will imply that F is a folding in some neighborhood of 0. Indeed, in view of (b), the Jacobian of f vanishes at all points $(x,0)$. Hence, for $y = 0$,

$$A_1(u) = \frac{\partial \tilde{v}(u,y)}{\partial y}|_{y=0} = J_F|_{y=0} = J_{f\circ\varphi^{-1}}|_{y=0} = 0.$$

By (10.4.8) and (10.4.9),

$$2A_2(0) = \frac{\partial^2 \tilde{v}(0,0)}{\partial y^2} = 2a > 0,$$

and by continuity, $A_2(u) > 0$ in $|u| < \delta$ for some $\delta > 0$.

Finally, since $A_2(u) > 0$ and $A_1(u) \equiv 0$ for $|u| < \delta$, we can get from (10.4.9)

$$F(u,y) = (u, \tilde{v}(u,y)) = (u, A_0(u) + w^2(u,y)),$$

where

$$w(u,y) = A_2^{1/2}(u)\left(y + \sum_{k=2}^{\infty} b_k(u) y^k\right)$$

for some functions $b_k(u)$. This shows that F is a local folding in some neighborhood of 0, and hence, so is f. □

10.4.2 Uniformization and Folds

Let μ be as in Sect. 10.4.1. Then local solutions in each of the domains D_1 and D_2 are related by conformal mappings and by the uniqueness theorems for folding solutions; this is the case for local folding solutions. One would expect that if (B) has a local folding solution at every point $z \in D \cap \mathbb{R}$, then (B) has a global folding solution in D. It turns out that this is not the case as shown in the following example.

10.4.3 Example

We will construct complex-valued measurable functions $A(z)$ and $B(z)$ in D, such that (B) has a local folding solution at every point in $D \cap \mathbb{R}$ and such that (B) has no global folding solution in D.

Let φ be a diffeomorphic map of D onto itself which maps $D \cap \mathbb{R}$ onto the arc $E = \{(x,y) \in \mathbb{R}^2 : y = \lambda x^2\} \cap D$ for some constant $\lambda > 0$. Since E is an analytic arc, there is a neighborhood U of $E, U \subset D$, and a conformal map $\psi : U \to \mathbb{C}$, such that $\psi(E) \subset \mathbb{R}$. Let $k : \psi(U) \to \mathbb{R}^2$ denote the standard folding map $k(x,y) = (x, |y|)$.

Now consider (B) with

$$A(z) = \begin{cases} \mu_\varphi(z), & z \in D_1 \\ 1, & z \in D \backslash D_1 \end{cases}$$

and

$$B(z) = \begin{cases} -1, & z \in D \backslash D_2 \\ -\bar{\mu}_\varphi(z), & z \in D_2 \end{cases},$$

where

$$\mu_\varphi(z) = \varphi_{\bar{z}} / \varphi_z, \qquad z \in D$$

is the complex dilatation of φ in D.

Clearly, $A(z)$ and $B(z)$ are measurable in D, and since φ is a diffeomorphism $\mu(z) = -A(z)/B(z) = \mu_\varphi(z)$ is locally bounded in D_1, and $1/\mu(z) = -B(z)/A(z) = \bar{\mu}_\varphi(z)$ is locally bounded in D_2. Furthermore, $k \circ \psi \circ \varphi : \varphi^{-1}(U) \to \mathbb{C}$ is a local folding solution of (B). Thus, (B) satisfies the assumptions in Theorem 10.5.

We will show that for any choice of $\lambda > 2$, (B) has no global folding solution in D. Fix $\lambda > 2$ and suppose that (B) has a global folding solution $g : D \to \mathbb{C}$. With no loss of generality, we may assume that $g(D \cap \mathbb{R}) \subset \mathbb{R}$ and $g(0) = 0$, since otherwise g can be replaced by another solution $h \circ g$ for some map h which is a solution in $g(D)$, conformal in int $g(D)$, and which maps $g(D \cap \mathbb{R})$ into $\mathbb{R}$ and $g(0)$ onto 0.

Now, by "unfolding" g, one obtains the homeomorphism $G : D \to \mathbb{C}$,

$$G(z) = \begin{cases} g(z), & z \in D \backslash D_2, \\ \bar{g}(z), & z \in D_2. \end{cases}$$

Then G and φ have the same complex dilatation $\mu(z) = \mu_\varphi(z)$ in D, and hence, $f = G \circ \varphi^{-1}$ is conformal in D and $f(0) = 0$. With no loss of generality, we may assume that $f'(0) = 1$. This can be obtained by replacing g by cg for some $c > 0$. Consequently, the map

$$f(z) = z + a_2 z^2 + \cdots$$

belongs to the class S of all normalized univalent maps in D, and hence, $|a_2| \leq 2$.

On the other hand, $f(E) \subset \mathbb{R}$, which implies

$$\mathrm{Im}\, f(t + \mathrm{i}\lambda t^2) = 0,$$

for all $t \in \mathbb{R}$, with $|t|$ sufficiently small, and straightforward computations give $|a_2| > \lambda$. This contradiction shows that (B) has no global folding solutions.

The following theorem shows that for some μ's as in Sect. 10.4.1, there is a local folding solution at every point of $D \cap \mathbb{R}$, yet any global solution must branch somewhere off $D \cap \mathbb{R}$:

Theorem 10.13. *[236]. Let D, D_1, D_2, and E be as in Sect. 10.4.1. Given a point $z_0 \in D \setminus E$, there exists μ as in Sect. 10.4.1 for which (B) has a local folding solution at every point of E, such that every global solution of (B) in D branches at z_0, and in particular, (B) has no (global) folding solution in D.*

In view of the last theorem and the fact that a local folding solution can be glued conformally, one may ask whether the existence of a global solution in D which folds along E and is locally injective off E implies that (B) has a global folding solution in D. The following theorem answers this question negatively:

Theorem 10.14. *[236]. Let D, D_1, D_2, and E be as in Sect. 10.4.1. There exists μ satisfying the condition of Sect. 10.4.1 such that (B) has a solution which folds along E and is locally injective off E, and (B) has no global folding solution in D.*

Appendix A
Archives of the Existence Theorems

Here, we give the original existence theorems for the Beltrami equations from the papers [56, 57, 98, 99, 158] and [174] as well as the original part of one example from [117] (cf. [116]), which have been important contributions to the development of the theory of the degenerate Beltrami equations. We will mainly retain the authors' styles of the aforementioned works. The corresponding advances can be found in Chap. 7; see historical comments and comparisons in Sect. 7.7.

A.1 The Existence Theorems of Brakalova–Jenkins

We start from the existence criteria in the papers of Brakalova–Jenkins [56, 57].

In the paper [56], they study the existence and uniqueness of solutions of the Beltrami equation $f_{\bar{z}}(z) = \mu(z) f_z(z)$, where $\mu(z)$ is a measurable function defined almost everywhere in a plane domain Δ with $\|\mu\|_\infty = 1$. Here, the partials $f_{\bar{z}}$ and f_z of a complex-valued function $f(z)$ exist almost everywhere. In case $\|\mu\|_\infty \le q < 1$, it is well known that homeomorphic solutions of the Beltrami equation are quasiconformal mappings. In case $\|\mu\|_\infty = 1$, much less is known. They give sufficient conditions on $\mu(z)$ that imply the existence of a homeomorphic solution of the Beltrami equation, which is ACL and whose partial derivatives $f_{\bar{z}}$ and f_z are locally in L^q for any $q < 2$. They also give uniqueness results. Their conditions improved already known results.

A.1.1 Introduction

In the Beltrami equation, (B) $\mu(z)$ is to be a measurable function defined almost everywhere in a plane domain Δ with $\|\mu\|_\infty = 1$. In case $\|\mu\|_\infty \le q < 1$, it is well known that homeomorphic solutions of the Beltrami equation are quasiconformal

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mappings with maximum dilatation $D(z) \leq K = (1+q)/(1-q)$. In case $\|\mu\|_\infty = 1$, much less is known. The only significant results known to the authors are due to Lehto [148, 149] and David [70].

In [148], Lehto treats the case of the plane with the following two stringent restrictions on $\mu(z)$:

(Λ_1) In the complement of a compact set of measure 0, $|\mu|$ is to be bounded from 1 on every compact subset.

(Λ_2) For any complex z and $0 < r_1 < r_2 < \infty$,

$$\int_{r_1}^{r_2} \left(1 + 2\pi \int_0^{2\pi} \frac{|1 - \mathrm{e}^{-2\mathrm{i}\theta}\mu(z + r\mathrm{e}^{\mathrm{i}\theta})|^2}{1 - |\mu(z + r\mathrm{e}^{\mathrm{i}\theta})|^2}\, \mathrm{d}\theta\right)^{-1} \frac{\mathrm{d}r}{r}$$

is strictly positive and tends to ∞ as $r_1 \to 0$ or $r_2 \to \infty$.

Under these conditions, he proves the existence of a homeomorphic solution to the Beltrami equation. Condition (Λ_2) essentially supposes the pointwise equicontinuity of the approximating functions to be discussed below, thus bypassing the most difficult step in the existence proof. In [148, 149], Lehto does not study the question of uniqueness.

In [70], David follows through the proof of existence in Ahlfors' monograph [9] giving some very detailed and complicated estimates. He considers the case when μ is defined in the plane and assumes the condition (A.1.1) that there exist constants $\alpha > 0$ and $C > 0$ such that for $\varepsilon > 0$ sufficiently small

$$\mathrm{mes}\{z : |\mu(z)| > 1 - \varepsilon\} \leq C\mathrm{e}^{-\frac{\alpha}{\varepsilon}}. \tag{A.1.1}$$

Under this condition, he proves the existence of a homeomorphic solution and shows that under suitable normalizations, it satisfies a uniqueness result.

The main results as well as various auxiliary results in this paper are obtained under conditions of the form

$$\int_B F\left(\frac{1}{1 - |\mu(z)|}\right) \mathrm{d}A < \Phi_B,$$

where B is a bounded measurable set, $\Phi_B > 0$ is a constant which depends on B, and $F(x)$, defined for $x \geq 1$, is either the identity function, or $F(x) = x^\lambda, \lambda > 1$, or $F(x) = \exp(x/(1 + \log x))$. With the choice $F(x) = \exp(x/(1 + \log x))$, i.e., condition (A.1.2) below, we prove the existence of a homeomorphic solution of the Beltrami equation, having properties detailed in the statement of our Theorem A.1 (existence theorem). We also give uniqueness results, which are stated as Theorems A.2 and A.3.

In the appendixes, we prove comparisons between our results and those already known. First, we show that David's results are not subsumed by Lehto's by providing an example where (A.1.1) holds while (Λ_1) does not. We also show that (A.1.1) implies conditions (A.1.2) and (A.1.3) of Theorem A.1, while our condition (A.1.2) does not imply David's condition (A.1.1).

A.1.2 Statements of the Main Results

Theorem A.1. ***(Existence Theorem).*** *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ with $\|\mu\|_\infty \leq 1$. For every bounded measurable set $B \subset \Delta$, let there exist a positive constant Φ_B such that*

$$\int_B \exp \frac{\dfrac{1}{1-|\mu|}}{1+\log\left(\dfrac{1}{1-|\mu|}\right)}\, \mathrm{d}A < \Phi_B, \tag{A.1.2}$$

and

$$\int_{\{|z|<R\}\cap\Delta} \frac{1}{1-|\mu|}\, \mathrm{d}A = O(R^2), \quad R \to \infty. \tag{A.1.3}$$

Then there exists a homeomorphic mapping $f(z)$ of Δ which is ACL *and whose partial derivatives f_z and $f_{\bar{z}}$ are in L^q on every compact subset of Δ for every $q < 2$ and which satisfies the Beltrami equation (B) a.e. The partials f_z and $f_{\bar{z}}$ are also distributional derivatives.*

Theorem A.2. ***(Uniqueness Theorem).*** *Let $\mu(z)$ be as in Theorem A.1, with Δ equal the plane. Let $\hat{f}(z)$ be a homeomorphism of the plane onto itself which has partial derivatives $\hat{f}_z(z)$ and $\hat{f}_{\bar{z}}(z)$ locally in L^2. If $\hat{f}$ satisfies the Beltrami equation (B) a.e., then*

$$\hat{f}(z) \equiv af(z) + b,$$

where a and b are constants, $a \neq 0$.

Theorem A.3. *If $\hat{f}$ is a homeomorphism of a domain Δ onto a domain Θ and has the same properties as in the above uniqueness theorem, then*

$$\hat{f}(z) = \xi(f(z)),$$

where ξ is a conformal mapping of $f(\Delta)$ onto Θ.

A.1.3 Construction of the Function f(z)

We can assume that $\mu(z)$ is defined in the plane by assigning the value 0 in the complement of Δ. For the construction of the function $f(z)$, we use only (A.1.2). Condition (A.1.3) is used to prove that $f(z)$ maps the plane onto itself.

Now, we define $\mu_n,\ n = 1, 2, \dots$ so that

$$\begin{aligned} \mu_n(z) &= \mu(z) \quad \text{if } |\mu(z)| \le 1 - 1/n \\ \mu_n(z) &= 0 \qquad \text{if } |\mu(z)| > 1 - 1/n. \end{aligned}$$

From the theory of quasiconformal mappings, we know that there exist q.c. mappings $f_n,\ n = 1, 2, \dots,$ of the plane onto itself with complex dilatations μ_n, $n = 1, 2, \dots$.

Let z_0 be a fixed point in the plane. For $r_2 > r_1 > 0$, denote by A the circular ring

$$A = \{z : r_1 < |z - z_0| < r_2\},$$

and by $M_n(r_1, r_2)$ the module of its image under f_n. The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation μ_n, where $\mu_n = \mu_n(z) = \mu_n(z_0 + re^{i\theta})$, in the following manner (see [6]):

$$M_n(r_1, r_2) \ge \int_{r_1}^{r_2} \frac{1}{\int\limits_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1 - |\mu_n|^2} d\theta} \frac{\mathrm{d}r}{r}.$$

Proposition A.1. *For any point z_0 and circular ring $A = \{r_1 < |z - z_0| < r_2\}$, the module $M_n(r_1, r_2)$ of the image of A under f_n tends uniformly to ∞ as $r_1 \to 0$.*

Proof. Using the lower estimate for the module of the image domain introduced earlier, we obtain

$$M_n(r_1, r_2) \ge \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int\limits_0^{2\pi} \frac{1}{1 - |\mu|} d\theta} \frac{\mathrm{d}r}{r}.$$

For any z_0 in a compact subset T of the plane containing the disc $|z - z_0| < r_2$,

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} \exp \frac{\frac{1}{1 - |\mu|}}{1 + \log \frac{1}{1 - |\mu|}} \mathrm{d}\theta \frac{\mathrm{d}r}{r} \le C,$$

where C depends only on the compact subset T and the choice of r_2. Now, we have

$$r^2 \int_0^{2\pi} \exp \frac{\frac{1}{1 - |\mu|}}{1 + \log \frac{1}{1 - |\mu|}} \mathrm{d}\theta < \frac{2C}{\log \frac{r_2}{r_1}}$$

on a set E of logarithmic measure $\frac{1}{2}\log\frac{r_2}{r_1}$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\frac{\frac{1}{1-|\mu|}}{1+\log\frac{1}{1-|\mu|}}\mathrm{d}\theta < \frac{C}{\pi r^2\log\frac{r_2}{r_1}} \quad \text{on } E.$$

The function

$$h(x) = \exp\frac{x}{1+\log x}$$

is convex and increasing, and $\lim_{x\to\infty} h(x) = \infty$. Let $l(y) = \log y \log\log y$ for $y > 1$ and let $h^{-1}(y)$ be the inverse of $h(x)$. One can show that

$$\lim_{x\to\infty}\frac{h^{-1}(h(x))}{l(h(x))} = 1,$$

so for some constant $c > 0$, $h^{-1}(y) < cl(y)$. Therefore, using the convexity of $h(x)$,

$$h\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}\mathrm{d}\theta\right) < \frac{C}{\pi r^2\log\frac{r_2}{r_1}} \quad \text{on } E$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}\mathrm{d}\theta < h^{-1}\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right) \quad \text{on } E.$$

Combined with the asymptotic behavior of $l(y)$, this implies

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}\mathrm{d}\theta < cl\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right) \quad \text{on } E.$$

Now, we fix R, $0 < R < 1$ and evaluate $M_n(R^{2^{N+1}}, R^{2^N})$, $N = 0, 1, \ldots$ for any fixed function f_n:

$$M_n\left(R^{2^{N+1}}, R^{2^N}\right) \geq \frac{1}{4}\int_{R^{2^{N+1}}}^{R^{2^N}}\frac{1}{\int\limits_0^{2\pi}\frac{1}{1-|\mu|}d\theta}\frac{\mathrm{d}r}{r} \geq \frac{1}{8\pi c}\int\limits_E\frac{1}{l\left(\frac{C}{\pi r^2\log(1/R)^N}\right)}\frac{\mathrm{d}r}{r}$$

$$\geq \frac{1}{8\pi c}\int_{R^{2^{N+1}}}^{R^{(3/2)2^N}}\frac{1}{l\left(\frac{C}{\pi r^2\log(1/R)^N}\right)}\frac{\mathrm{d}r}{r}.$$

Using the properties of $l(y)$, the logarithmic function and explicit integration, one can prove that

$$\frac{1}{8\pi c}\int_{R^{2^{N+1}}}^{R^{(3/2)2^N}} \frac{1}{l\left(\frac{C}{\pi r^2 \log(1/R)^N}\right)}\frac{\mathrm{d}r}{r} \sim \frac{A_0}{N},\ A_0>0, \quad \text{as } N\to\infty.$$

Thus, for some $L>0$

$$\sum_{N=L}^{\infty} M_n\left(R^{2^{N+1}}, R^{2^N}\right) > \sum_{N=L}^{\infty}\frac{A_1}{N},$$

A_1 being some constant. By the additive property of the module, it follows that

$$\lim_{N\to\infty} M_n(R^{2^N}, R) = \infty,$$

independent of the choice of z_0 and n.

Now, let $0<R<\min\{1,r_2\}$. For any fixed N, there is r_1, sufficiently small, such that

$$\left\{R^{2^N} < |z-z_0| < R\right\} \subset \{r_1 < |z-z_0| < r_2\},$$

and therefore, $M_n(r_1,r_2) \geq M_n(R^{2^N},R)$ or $\lim_{r_1\to 0} M_n(r_1,r_2) \geq M_n(R^{2^N},R)$ for any fixed N. Thus, $\lim_{r_1\to 0} M_n(r_1,r_2) = \infty$ independent of the choice of z_0 and n. □

From now on, we will assume that the quasiconformal mappings $\{f_n(z)\}$, $n=1,2,\ldots,$ have two fixed points a_1 and a_2 with $d=|a_1-a_2|>0$.

Proposition A.2. *The family of quasiconformal mappings* $\{f_n(z)\}$, $n=1,2,\ldots,$ *is uniformly equicontinuous on each compact subset* T *of the plane.*

Proof. Suppose that this is not the case. Then for some $\varepsilon>0$ and every $\delta>0$, $0<\delta<d/2$, there exist f_n and points $z_1^{(n)}$ and $z_2^{(n)}$ in T, such that $|z_1^{(n)}-z_2^{(n)}|<\delta$ and $|f_n(z_1^{(n)})-f_n(z_2^{(n)})|\geq\varepsilon$.

Denote by z_0 an accumulation point of the sequence $\{z_1^{(n)}\}$. The distance between z_0 and at least one of the points a_1 and a_2 is at least $d/2$. We assume that this point is a_1. The image of the ring $\delta<|z-z_0|<d/2$ lies in a doubly connected domain for which one complementary component has a diameter $\geq\varepsilon$, and the other contains a_1 and ∞, thus the module of the image will be finite, which contradicts Proposition A.1. □

Proposition A.3. *For the sequence* $f_n(z)$, *there exists a subsequence of functions which converges uniformly to a function* $f(z)$ *on compact subsets.*

Proof. The result follows from Proposition A.2 and Arzela–Ascoli's theorem. □

We denote the converging sequence again by $f_n(z)$.

Proposition A.4. *If*

$$\int_{|z|<R} \frac{1}{1-|\mu|}\,\mathrm{d}A = O(R^2) \qquad \text{as } R \to \infty,$$

then $f_n(z)$ converges uniformly to ∞, as $z \to \infty$.

Proof. There exists a constant P such that

$$\int\int_{R \le |z| \le R^2} \frac{1}{1-|\mu|}\,\mathrm{d}A \le PR^4.$$

We can show that for $R \ge R_0 > 0$,

$$\int_0^{2\pi} \frac{1}{1-|\mu|}\,\mathrm{d}\theta \le \frac{3PR^2}{r}$$

on a set E, $E \subset (R, R^2)$, of length at least $(R^2 - R)/2$. Otherwise, there would be a set X of length equal to $(R^2 - R)/2$ such that

$$\int_R^{R^2}\int_0^{2\pi} \frac{1}{1-|\mu|} r\mathrm{d}r\mathrm{d}\theta \ge 2\pi \int_X \frac{3PR^2}{r} r\mathrm{d}r \ge 3\pi PR^2(R^2 - R),$$

which contradicts our assumption. Therefore,

$$\int_R^{R^2} \frac{1}{\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta} \frac{\mathrm{d}r}{r} \ge \frac{R^2 - R}{6PR^2}.$$

Now, we fix $R > R_0 > 1$. There exists P such that

$$\int\int_{R^{k-1}<|z|<R^k} \frac{\mathrm{d}A}{1-|\mu|} \le PR^{4k}, \qquad \text{for} \quad k = 1, 2, \ldots.$$

Let A_N be the annulus $\{z : R < |z| < R^{2N}\}$ and let M_n^N be the module of its image under f_n, $n = 1, 2, \ldots$. Then

$$\begin{aligned} M_n^N &\ge \int_R^{R^{2N}} \frac{1}{\int_0^{2\pi} \frac{|1-\mathrm{e}^{-2i\theta}\mu_n|^2}{1-|\mu_n|^2}\mathrm{d}\theta} \frac{\mathrm{d}r}{r} \ge \frac{1}{4}\sum_{k=1}^{N} \int_{R^{2(k-1)}}^{R^{2k}} \frac{1}{\int_0^{2\pi} \frac{1}{1-|\mu|}\mathrm{d}\theta} \frac{\mathrm{d}r}{r} \\ &\ge \frac{1}{4}\sum_{k=1}^{N} \frac{R^{2k} - R^{2(k-1)}}{6PR^{2k}} = \frac{(R^2-1)}{24PR^2} N. \end{aligned}$$

Therefore, M_n^N tends uniformly to ∞ as $N \to \infty$. We know also that as $n \to \infty$, $f_n(z)$ tends uniformly on compact subsets to $f(z)$, so we can conclude that

$$f(z) \to \infty, \qquad \text{as} \quad z \to \infty. \qquad \square$$

A.1.4 f(z) Is a Homeomorphism

Lemma A.1. *Let h_n, $n = 1, 2, \ldots$, be a sequence of homeomorphisms of the plane onto itself with two fixed points a_1 and a_2. If h_n converges uniformly on every compact subset of the plane to a function $h(z)$, and if for every annulus A there is a $q > 0$ such that*

$$M(h_n(A)) \geq q, \tag{A.1.4}$$

then h is a homeomorphism.

Proof. Assume that h is not a homeomorphism. Then there exist two points $z_1 \neq z_2$ such that $h(z_1) = h(z_2) = w_0$. We can assume that none of the points z_1, z_2, and w_0 coincide with a_2 and that z_1 does not lie on the segment connecting z_2 and a_2. Construct a line t through z_1 which does not meet that segment, a circle C_1 passing through a_2 and z_2 which does not meet t, and a concentric circle C_2, outside of C_1, which also does not meet t. Let D be the doubly connected domain bounded by C_1 and t and A be the ring domain bounded by C_1 and C_2. Then $A \subset D$ and

$$M(h_n(D)) > M(h_n(A)).$$

On the other hand, the spherical diameters of $h_n(C_1)$ and $h_n(t)$ are bounded away from zero, while the spherical distance between them tends to 0 as $n \to \infty$. This implies that $M(h_n(D)) \to 0$ as $n \to \infty$, which contradicts (A.1.4). $\square$

Lemma A.2. *Let μ^* be a measurable complex-valued function with $\|\mu^*\|_\infty = 1$ and let μ_n^* be a sequence of measurable functions, constructed in the same way as in Sect. A.1.3. If:*

(i) h_n is the corresponding quasiconformal mappings of the complex plane onto itself with two distinct fixed points,
(ii) h_n converges uniformly on compact subsets of the plane to h,
(iii) For each compact subset L in the plane, and some positive constant Φ_L,

$$\int_L \frac{1}{1 - |\mu^*|} \mathrm{d}A < \Phi_L, \tag{A.1.5}$$

then:

1. *There exists* $q>0$ *such that* $M(h_n(A))\geq q$.
2. h *is a homeomorphism.*

Proof. Fix a point z_0 in the plane, and for $0<r_1<r_2$, let S be the annulus $\{z: r_1\leq |z-z_0|\leq r_2\}$. According to (A.1.5), there exists constant Φ_S such that

$$\int_S \frac{1}{1-|\mu^*|}\mathrm{d}A<\Phi_S. \tag{A.1.6}$$

As in Proposition A.1, using the appropriate estimate for the module of the image

$$M(h_n(A))\geq \pi\int_{r_1}^{r_2}\frac{\mathrm{d}r/r}{\int_0^{2\pi}\frac{1}{1-|\mu^*|}\mathrm{d}\theta}.$$

There is a measurable subset X of $[r_1,r_2]$ of measure $(r_2-r_1)/2$ on which

$$\int_0^{2\pi}\frac{1}{1-|\mu^*|}\mathrm{d}\theta\leq \frac{8\Phi_S}{(r_2-r_1)(r_2+3r_1)}.$$

Otherwise, we would have

$$\int_{r_1}^{r_2}\int_0^{2\pi}\frac{1}{1-|\mu^*|}\mathrm{d}\theta r\mathrm{d}r\geq \frac{8\Phi_S}{(r_2-r_1)(r_2+3r_1)}\int_{r_1}^{(r_1+r_2)/2} r\mathrm{d}r=\Phi_S,$$

which is a contradiction to (A.1.3).

Therefore,

$$M(h_n(A))\geq \pi\int_X\frac{\mathrm{d}r/r}{\int_0^{2\pi}\frac{1}{1-|\mu^*|}\mathrm{d}\theta}\geq \frac{\pi(r_2+3r_1)(r_2-r_1)}{8\Phi_S}\int_{(r_1+r_2)/2}^{r_2}\frac{\mathrm{d}r}{r}=q>0.$$

By Lemma A.1, $h(z)$ is a homeomorphism. □

Proposition A.5. *The function f(z) constructed in Proposition A.3 is a homeomorphism and maps the plane onto the plane.*

Proof. Inequality (A.1.5) for μ follows from (A). This together with the properties of $f_n(z)$ implies that $f(z)$ is a homeomorphism. □

A.1.5 Differentiability Properties of f(z)

Lemma A.3. *If $h_n(z)$ is quasiconformal mappings of the plane onto itself with complex dilatation μ_n^*, satisfying $|\mu_n^*| \leq |\mu^*| \leq 1$ a.e., if $h_n(z)$ converges uniformly on the compact subsets of the plane to $h(z)$, and if for every rectangle R there is a positive constant Φ_R such that*

$$\int_R \frac{1}{1-|\mu^*|}\,\mathrm{d}A < \Phi_R, \tag{A.1.7}$$

then $h(z)$ is ACL.

Proof. Let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. Denote by I_y the segment with ordinate y and $a \leq x \leq b$ and by R_y the rectangle which is the subset of R below that segment. Let $A(y)$ be the area of the image of R_y under $h(z)$. Now, $A(y)$ is an increasing function of y and thus has a finite derivative $A'(y)$ for almost all y in the interval $[c,d]$. Moreover, by Fubini's theorem,

$$\int_{I_y} \frac{1}{1-|\mu^*|}\,\mathrm{d}x$$

is finite for almost all y in $[c,d]$. Further, for any systems of infinitely many nonoverlapping subintervals (ξ_k, ξ_k^*), $k = 1,2,\ldots,m$, of (a,b) and any $\delta > 0$ sufficiently small (i.e., $y+\delta < d$), consider

$$\frac{1}{\delta}\sum_{k=1}^{m}\int_{\xi_k}^{\xi_{k+1}}\int_{y}^{y+\delta}\frac{1}{1-|\mu^*|}\,\mathrm{d}A.$$

By the theorem of Lebesgue, for almost all y in (c,d), this expression has a limit as $\delta \to 0$, which is

$$\sum_{k=1}^{m}\int_{\xi_k}^{\xi_{k+1}}\frac{1}{1-|\mu^*|}\,\mathrm{d}A.$$

Allowing all possible rational pairs of ξ_k, ξ_k^*, this will hold simultaneously for almost all y in (c,d). Choose y_0 for which all of this applies. Now, let (x_k, x_k^*) be an arbitrary system of nonoverlapping subintervals of (a,b). Let ξ_k, ξ_k^* be rational such that $x_k < \xi_k < \xi_k^* < x_k^*$, and let

$$w_k = h(x_k + \mathrm{i}y_0), \qquad w_k^* = h(x_k^* + \mathrm{i}y_0),$$
$$\omega_k = h(\xi_k + \mathrm{i}y_0), \qquad \omega_k^* = h(\xi_k^* + \mathrm{i}y_0).$$

To prove that f is absolutely continuous on I_{y_0}, it is enough to show that $\sum_{k=1}^{m}|w_k^* - w_k|$ has an upper bound which tends to zero with $\sum_{k=1}^{m}(x_k^* - x_k)$.

We choose $\delta > 0$ with $y_0 + \delta < d$ and denote by $R_k(\delta)$ the rectangle $\{(x,y) : \xi_k < x < \xi_k^*,\ y_0 < y < y_0 + \delta\}$. Assuming its corners as vertices, it becomes a quadrangle whose image under $h(z)$ we denote by $Q_k(\delta)$. Let $m(Q_k(\delta))$ be the area of $Q_k(\delta)$ and $\beta_k(\delta)$ the distance between the images of the vertical sides of $R_k(\delta)$. Let $M(Q_k(\delta))$ be the module of $Q_k(\delta)$ for curves joining the images of horizontal sides of $R_k(\delta)$. Then

$$M(Q_k(\delta)) \geq \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))}.$$

Let the corresponding entities under $h_j(z)$ be $R_k^j(\delta)$, $Q_k^j(\delta)$, and $M(Q_k^j(\delta))$. By an upper estimate for the module of a quadrangle from [6], after applying the Schwarz inequality, we have

$$M(Q_k^j(\delta)) \leq \frac{1}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{1+|\mu_j^*|^2}{1-|\mu_j^*|^2}\,\mathrm{d}A \leq \frac{2}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|}\,\mathrm{d}A.$$

Passing to the limit as $j \to \infty$

$$M(Q_k(\delta)) \leq \frac{2}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|}\,\mathrm{d}A,$$

and therefore,

$$\frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \leq \frac{2}{\delta^2} \int_{R_k(\delta)} \frac{1}{1-|\mu^*|}\,\mathrm{d}A.$$

Thus,

$$\sum_{k=1}^{m} \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \leq \frac{2}{\delta^2} \int_{\cup R_k(\delta)} \frac{1}{1-|\mu^*|}\,\mathrm{d}A.$$

Using the Schwarz inequality,

$$\begin{aligned}\left(\sum_{k=1}^{m} \beta_k(\delta)\right)^2 &\leq \sum_{k=1}^{m} \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \cdot \sum_{k=1}^{m} m(Q_k(\delta)) \\ &\leq \frac{2}{\delta^2} \int\int_{\cup R_k(\delta)} \frac{1}{1-|\mu^*|}\,\mathrm{d}A \cdot (A(y_0+\delta) - A(y_0)).\end{aligned}$$

Letting $\delta \to 0$, we have

$$\left(\sum_{k=1}^{m} |\omega_k^* - \omega_k|\right)^2 \leq 2A'(y_0) \sum_{k=1}^{m} \int_{\xi_k}^{\xi_k^*} \frac{1}{1-|\mu^*|}\, \mathrm{d}x$$

$$\leq 2A'(y_0) \sum_{k=1}^{m} \int_{x_k}^{x_k^*} \frac{1}{1-|\mu^*|}\, \mathrm{d}x.$$

Letting $\xi_k \to x_k$, $\xi_k^* \to x_k^*$, we have

$$\left(\sum_{k=1}^{m} |w_k^* - w_k|\right)^2 \leq 2A'(y_0) \sum_{k=1}^{m} \int_{x_k}^{x_k^*} \frac{1}{1-|\mu^*|}\, \mathrm{d}x.$$

Since the integral is absolutely continuous as a set function, this completes the proof. □

Proposition A.6. *The function f(z) is* ACL.

Proof. Since (A) holds for μ, so does (A.1.7), and $\mu_n, \mu, f_n,$ and f satisfy the conditions of Lemma A.3. This implies that f is ACL. □

Lemma A.4. *If:*

(i) h_n *is quasiconformal mappings of the plane onto itself with complex dilatation* μ_n^* *satisfying* $|\mu_n^*| \leq |\mu^*| \leq 1$ *a.e.,*
(ii) $h_n(z)$ *converges uniformly on compact subsets of the plane to* $h(z)$*,*
(iii) For some $\lambda > 1$ *and any compact subset* L *of the plane there is a positive constant* Φ_L *such that*

$$\int_L \left(\frac{1}{1-|\mu^*|}\right)^{\lambda} \mathrm{d}A \leq \Phi_L, \tag{A.1.8}$$

then $h(z)$ *has partial derivatives* h_z *and* $h_{\bar z}$ *which are in* L^q *on every compact subset of the plane, where* $q \leq (2\lambda)/(1+\lambda)$*.*

Proof. We consider a Jordan domain Θ such that L is contained in it. By condition (ii), h_n has a subsequence converging uniformly on $\bar\Theta$ which we again denote by h_n. Let J_n denote the Jacobian of h_n. We will use the fact that a.e.

$$|(h_n)_z| = \frac{(J_n)^{1/2}}{(1-|\mu_n^*|^2)^{1/2}}.$$

Using the Hölder inequality, it follows that

$$\int_{\bar{\Theta}} |(h_n)_z|^q \mathrm{d}A = \int_{\bar{\Theta}} \frac{(J_n)^{q/2}}{(1-|\mu_n^*|^2)^{q/2}} \mathrm{d}A$$
$$\leq \left(\int_{\bar{\Theta}} (J_n)^{\frac{qp'}{2}} \mathrm{d}A \right)^{\frac{1}{p'}} \left(\int_{\bar{\Theta}} \frac{1}{(1-|\mu_n^*|^2)^{\frac{qq'}{2}}} \mathrm{d}A \right)^{\frac{1}{q'}},$$

where $1/p' + 1/q' = 1$. We choose $p' = 2/q$, which implies that q is at most $2\lambda/(1+\lambda)$. Then the first integral becomes $\int_{\bar{\Theta}} J_n \ \mathrm{d}A$, which is the area of the image of $\bar{\Theta}$ under $h(z)$ and therefore is finite. Thus, the first integral is uniformly bounded for all n. The second integral is also uniformly bounded because of (A.1.8). Therefore,

$$\int_{\bar{\Theta}} |(h_n)_z|^q \mathrm{d}A$$

is uniformly bounded for all n. For $q > 1$, $L^q(\bar{\Theta})$ is a Banach space. Therefore, the sequence of functions $(h_n)_z$ has a weakly convergent subsequence $(h_{n_k})_z$. The same is true for $(h_n)_{\bar{z}}$ because h_n is sense preserving, and thus, $|(h_n)_{\bar{z}}| < |(h_n)_z|$. The same is true for the real partials $(h_{n_k})_x$ and $(h_{n_k})_y$, which converge to $\hat{h}$ and $\hat{\hat{h}}$ in $L^q(\bar{\Theta})$. Since h_n is quasiconformal, $(h_{n_k})_x$ and $(h_{n_k})_y$ are distributional derivatives; see [9], p. 28. For any test function $t(x,y)$

$$\int_{\bar{\Theta}} (h_{n_k})_x t(x,y) \mathrm{d}A = -\int_{\bar{\Theta}} h_{n_k} t(x,y)_x \mathrm{d}A.$$

Passing to the limits as $k \to \infty$, by the weak convergence of $(h_{n_k})_x$ and the uniform convergence of h_{n_k}, we have

$$\int_{\bar{\Theta}} \hat{h} t \ \mathrm{d}A = -\int_{\bar{\Theta}} h t_x \ \mathrm{d}A.$$

Similarly,

$$\int_{\bar{\Theta}} \hat{\hat{h}} t \ \mathrm{d}A = -\int_{\bar{\Theta}} h t_y \ \mathrm{d}A.$$

Thus, $h(z)$ has distributional derivatives in $L^q(\bar{\Theta})$. By Theorem 2.1.4 [272], h, being continuous, coincides with its representative in the context of this theorem and its partial $h_z \in L^q(\bar{\Theta})$. The same is true for $h_{\bar{z}}$, and therefore, both partials are in $L^q(L)$.

Note that this also proves that h is ACL, but under stronger conditions than in Lemma A.3. □

Proposition A.7. *The partials f_z and $f_{\bar{z}}$ of $f(z)$ are in L^q on compact subsets of the plane for every $q < 2$.*

Proof. Since (A) holds for μ, so does (A.1.8) for every $\lambda > 1$. This together with the properties of f_n, f, μ_n, and μ implies that f_z and $f_{\bar{z}}$ are in $L^q(L)$ for every compact set L in the plane and $1 < q < 2$, and thus, for any q, $0 < q < 2$. □

A.1.6 The Limit Function Satisfies the Beltrami Equation

Lemma A.5. *Let h_n be quasiconformal mappings of the plane onto itself with complex dilatation μ_n^*, satisfying $|\mu_n^*| < |\mu^*| \le 1, \lim_{n\to\infty} \mu_n^* = \mu^*$, a.e. Let h_n converge uniformly on compact subsets of the plane to h and let $\lambda > 1$. If, for any rectangle R with sides parallel to the axes, there exists a positive constant Φ_R such that*

$$\int_R \left(\frac{1}{1-|\mu^*|}\right)^{\lambda} \mathrm{d}A \le \Phi_R, \tag{A.1.9}$$

then

$$h_{\bar{z}}(z) = \mu^* h_z(z) \qquad a.e.$$

Proof. Denote $\zeta(z) = h_{\bar{z}}(z) - \mu^*(z) h_z(z)$; $\zeta(z)$ is defined a.e. Now, we can write

$$\zeta = [h_{\bar{z}} - (h_n)_{\bar{z}}] + [(h_n)_{\bar{z}} - \mu_n^*(h_n)_z] + [\mu^*(h_n)_z - \mu_n^*(h_n)_z] + [\mu^*(h_n)_z - \mu^* h_z].$$

If we set

$$\begin{aligned} I_{1,n}(z) &= h_{\bar{z}} - (h_n)_{\bar{z}}, \\ I_{2,n}(z) &= (h_n)_{\bar{z}} - \mu_n^*(h_n)_z, \\ I_{3,n}(z) &= \mu_n^*(h_n)_z - \mu^*(h_n)_z, \\ I_{4,n}(z) &= \mu^*(h_n)_z - \mu^* h_z, \end{aligned}$$

then

$$\int_R \zeta \,\mathrm{d}A = \int_R I_{1,n} \,\mathrm{d}A + \int_R I_{2,n} \,\mathrm{d}A + \int_R I_{3,n} \,\mathrm{d}A + \int_R I_{4,n} \,\mathrm{d}A.$$

The integral $\int_R I_{2,n} \mathrm{d}A = 0$ because $I_{2,n} = 0$ a.e. According to Lemma A.4, h has L^1 derivatives in R and so do h_n as quasiconformal mappings. By formula (6.17) on p. 50 of [152],

$$\int_R \left[(h_n)_{\bar{z}} - h_{\bar{z}}\right] \mathrm{d}A = \frac{1}{2\mathrm{i}} \int_{\partial R} \left[h_n - h\right] \mathrm{d}z.$$

Since h_n tends uniformly to h on the boundary ∂R, the above integrals converge to 0 as $n \to \infty$. So we have

$$\lim_{n\to\infty} \int_R I_{1,n}\, \mathrm{d}A = 0. \tag{A.1.10}$$

Further, from Hölder's inequality, it follows that

$$\begin{aligned}\left|\int_R I_{3,n}\, \mathrm{d}A\right| &= \left|\int_R (\mu_n^* - \mu^*)(h_n)_z\, \mathrm{d}A\right| \\ &\leq \left(\int_R |\mu_n^* - \mu^*|^p \mathrm{d}A\right)^{\frac{1}{p}} \left(\int_R |(h_n)_z|^q \mathrm{d}A\right)^{\frac{1}{q}},\end{aligned}$$

where $1/p + 1/q = 1$, p, $q > 0$, and $q = 2\lambda/(1+\lambda)$; thus $1 < q < 2$. As in the proof of Lemma A.4, one can show that $\int_R \left|(h_n)_z\right|^q \mathrm{d}A$ are uniformly bounded for all n. Since $\mu_n^* \to \mu^*$ as $n \to \infty$ a.e., and $\left|\mu_n^* - \mu^*\right|^p \leq 2^p$, by Lebesgue's theorem

$$\int_R |\mu_n^* - \mu^*|^p \mathrm{d}A \to 0 \qquad \text{as } n \to \infty.$$

Therefore,

$$\iint_R I_{3,n}\, \mathrm{d}A \to 0 \qquad \text{as } n \to \infty.$$

By p. 138 of [152], μ^* can be approximated by uniformly bounded step functions ϕ_k with constant values on squares with sides parallel to the axes, so that $\lim_{k\to\infty} \phi_k = \mu^*$ a.e. in R. Using this, we have

$$\int_R I_{4,n}\, \mathrm{d}A = \int_R (\mu^* - \phi_k)\big((h_n)_z - h_z\big)\, \mathrm{d}A + \int_R \phi_k\big((h_n)_z - h_z\big)\, \mathrm{d}A.$$

By Hölder's inequality,

$$\int_R \left|(\mu^* - \phi_k)\big((h_n)_z - h_z\big)\right| \mathrm{d}A \leq \left(\int_R |\mu^* - \phi_k|^p \mathrm{d}A\right)^{\frac{1}{p}} \left(\int_R |(h_n)_z - h_z|^q \mathrm{d}A\right)^{\frac{1}{q}},$$

with q as above. Since $\int_R \left|(h_n)_z\right|^q \mathrm{d}A$ is uniformly bounded and $h_z \in L^q(R)$, then $\int_R \left|(h_n)_z - h_z\right|^q \mathrm{d}A$ is uniformly bounded. Since $\lim_{k\to\infty} \phi_k = \mu^*$ a.e., by Lebesgue's theorem, we can choose k_0 so that for $k > k_0$ and for every n,

$$\int_R |\mu^* - \phi_k|\left|(h_n)_z - h_z\right| \mathrm{d}A \leq \varepsilon.$$

On the other hand, since ϕ_{k_0} has a constant value on the finitely many squares or parts of squares covering R and by the argument used to prove (A.1.10), we have

$$\int_R \phi_{k_0}\left((h_n)_z - h_z\right) \mathrm{d}A \to 0 \qquad \text{as } n \to \infty.$$

Therefore, there is N_0 so that for $n \geq N_0$

$$\left| \int_R \phi_{k_0}\left((h_n)_z - h_z\right) \mathrm{d}A \right| < \varepsilon$$

or

$$\left| \int_R I_{4,n} \,\mathrm{d}A \right| \leq 2\varepsilon.$$

Thus,

$$\int_R I_{4,n} \,\mathrm{d}A \to 0 \qquad \text{as } n \to \infty,$$

and

$$\int_R \zeta \,\mathrm{d}A = 0.$$

Using the measure-theoretic argument on p. 189 of [152], one can show that $\zeta = 0$ a.e., and hence, we have

$$h_{\bar{z}}(z) - \mu^*(z) h_z(z) = 0 \qquad \text{a.e.}$$

□

Proposition A.8. *The function $f(z)$ satisfies the Beltrami equation (B).*

Proof. Since (A.1.2) holds for μ, so does (A.1.9). This together with the properties of f_n, f, μ_n, and μ implies that f satisfies the Beltrami equation (B). □

A.1.7 The Inverse Function g(w) of f(z)

Lemma A.6. *Let:*

(i) h_n, $n = 1, 2, \ldots$, be homeomorphisms of the plane onto itself with fixed points $a_1 \neq a_2$.
(ii) h_n converge uniformly to h on compact subsets of the plane.
(iii) h_n converge uniformly to ∞ as $z \to \infty$ and $n \to \infty$.

(iv) For any fixed annulus A, the module $M(h_n(A))$ be uniformly bounded away from zero.
(v) l_n, $n = 1, 2, \ldots$, and l be the inverse homeomorphisms to h_n, $n = 1, 2, \ldots$, and h.

Then the sequence l_n, $n = 1, 2, \ldots$, is pointwise equicontinuous, and l_n converges uniformly to l on compact subsets of the plane.

Proof. Assume that the sequence l_n is not pointwise equicontinuous. Then there exist a point p and a number $\varepsilon > 0$, a sequence $\delta_k \downarrow 0$, sequences $\{w_k\}_{k=1}^{\infty}$ and $\{l_{n_k}\}_{k=1}^{\infty}$ such that $|w_k - p| < \delta_k$ and

$$\left|l_{n_k}(w_k) - l_{n_k}(p)\right| \geq \varepsilon. \tag{A.1.11}$$

The sequence $l_{n_k}(p)$ has a finite accumulation point because h_n converges uniformly to ∞ as $z \to \infty$ and $n \to \infty$. We can find a subsequence which tends to this accumulation point. We denote this subsequence again by $\{l_{n_k}\}_{k=1}^{\infty}$ and the accumulation point by z_0. If necessary, after taking another subsequence, we can assert that there is a finite point q such that $l_{n_k}(p) \to q$, as $k \to \infty$. Because of (A.1.11), we have $z_0 \neq q$.

Assume that $z_0 \neq a_1$ and that q does not lie on the segment joining z_0 and a_1. Draw a circle Γ_1^0 through z_0 and a_1 such that q is outside of Γ_1^0 and a line Γ_2^0 through q not meeting Γ_1^0. Draw a second circle Γ_1 tangent to Γ_1^0 at a_1 of larger radius which does not meet Γ_2^0 and a line Γ_2 parallel to Γ_2^0 which lies between Γ_1 and Γ_2^0. There is an integer k_0 such that for $k \geq k_0$, $l_k(w_k)$ is inside Γ_1 and $l_{n_k}(p)$ is in the half plane determined by Γ_2 not containing Γ_1. For each $k \geq k_0$, we construct a circle $\Gamma_1^{(k)}$ passing through a_1 and $l_{n_k}(w_k)$, tangent to Γ_1 at a_1, and a line $\Gamma_2^{(k)}$ parallel to Γ_2 and passing through $l_{n_k}(p)$.

For each $k \geq k_0$, $h_{n_k}(\Gamma_1^{(k)})$ passes through a_1 and w_k, and $h_{n_k}(\Gamma_2^{(k)})$ passes through p and ∞. Therefore, the spherical diameter of each of the families of curves $h_{n_k}(\Gamma_1^{(k)})$ and $h_{n_k}(\Gamma_2^{(k)})$ is bounded away from 0. However, the spherical distance between $\Gamma_1^{(k)}$ and $\Gamma_2^{(k)}$ is less than δ_k. If we denote by A_k the doubly connected domain bounded by $\Gamma_1^{(k)}$ and $\Gamma_2^{(k)}$ and by $M\big(h_{n_k}(A_{n_k})\big)$ the module of its image under h_{n_k}, it follows from Lemma 2 of [148] that $M\big(h_{n_k}(A_k)\big) \to 0$ as $k \to \infty$.

Now, consider A to be the doubly connected domain bounded by the curves Γ_1 and Γ_2, and let $M\big(h_{n_k}(A)\big)$ be the module of its image under h_{n_k}. Since $A \subset A_k$, for $k \geq k_0$

$$M\big(h_{n_k}(A)\big) \leq M\big(h_{n_k}(A_k)\big),$$

and thus, $M\big(h_{n_k}(A)\big) \to 0$ as $k \to \infty$. This contradicts our assumption (iv).

If q lies on the segment joining z_0 and a_1 and $q \neq a_1$, we reverse the roles of z_0 and q in the above proof. If $q = a_1$ and $a_2 \neq z_0$, then the proof proceeds as above interchanging the roles of a_1 and a_2. Finally, if $q = a_1$ and $z_0 = a_2$, instead of taking circles tangent to a_1, we take circles centered at a_1.

This proves the pointwise equicontinuity of the sequence $\{l_n\}$. Therefore, by Lemma 5.1, p. 71 of [152], there exists a subsequence of $\{l_{n_k}\}_{k=1}^{\infty}$ converging uniformly on compact subsets of the plane to a continuous function l_0. For any z and k, $h_{n_k}\big(l_{n_k}(z)\big) = z$ and $l_{n_k}\big(h_{n_k}(w)\big) = w$. Since h_{n_k} converges uniformly to h and l_{n_k} converges uniformly to l_0, we have

$$h\big(l_0(z)\big) = z \qquad \text{and} \qquad l_0\big(h(w)\big) = w.$$

Therefore, $l_0(z)$ is identical to $l(z)$, and the sequence $l_n(z)$ converges to $l(z)$ uniformly on compact subsets of the plane. □

Let $g(z)$ be the inverse of $f(z)$, and let $g_n(z)$ be the inverse of $f_n(z)$, for $n = 1,2,\ldots$.

Proposition A.9. *The sequence g_n converges uniformly to g on compact subsets of the plane.*

Proof. Now, since (A) holds for f_n, then (A.1.5) holds, and by Lemma A.2, $M\big(f_n(A)\big)$ is uniformly bounded away from 0. This together with the properties of f_n, f, the definition of g_n and g, and Lemma A.6 implies the statement of the above proposition. □

Proposition A.10. *The function g is* ACL *and g_w is locally in L^2.*

Proof. Let B be a compact set in the plane and U be a compact disk such that B lies in the interior of U. Let J_n denote the Jacobian of f_n. The functions $g_n,\ n = 1,2,\ldots$, are quasiconformal with Jacobian $1/(J_n\big(g_n(w)\big))$. We denote their complex dilatation by $\nu_n,\ n = 1,2,\ldots$. Thus,

$$\int_U \big|(g_n)_w(w)\big|^2 \mathrm{d}A_w = \int_U \frac{1}{J_n\big(g_n(w)\big)\big(1-\big|\nu_n(w)\big|^2\big)}\mathrm{d}A_w$$
$$= \int_{g_n(U)} \frac{J_n(z)\mathrm{d}A_z}{J_n(z)\big(1-\big|\mu_n(z)\big|^2\big)} = \int_{g_n(U)} \frac{1}{1-\big|\mu_n(z)\big|^2}\mathrm{d}A_z.$$

By the uniform convergence of g_n, all $g_n(U)$ lie in a compact set V, so

$$\int_{g_n(U)} \frac{1}{1-\big|\mu_n(z)\big|}\mathrm{d}A_z \le \int_V \frac{1}{1-\big|\mu(z)\big|}\mathrm{d}A_z,$$

and all the terms $\int_U \big|\big(g_n(w)\big)_w\big|^2 \mathrm{d}A_w$ are uniformly bounded because (A) holds for $\mu(z)$. Since $g_n,\ n = 1,2,\ldots$, is sense preserving, $\int_U \big|\big(g_n(w)\big)_{\bar{w}}\big|^2 \mathrm{d}A_w$ is also uniformly bounded. Thus, the sequences $(g_n)_w$ and $(g_n)_{\bar{w}}$ are uniformly bounded sequences in the Hilbert space $L^2(U)$, and there are subsequences $(g_{n_k})_w$ and $(g_{n_k})_{\bar{w}}$ which converge weakly in $L^2(U)$. The same is true for the real partials $(g_{n_k})_u$

and $(g_{n_k})_v$, $w = u + \mathrm{i}v$, which converge weakly to $\hat{g}$ and $\hat{\hat{g}}$ in $L^2(U)$. Since g_n are quasiconformal, $(g_{n_k})_u$ and $(g_{n_k})_v$ are distributional derivatives (see p. 28 of [9]). Then for any test function $t(u,v)$,

$$\int_U (g_{n_k})_u t \,\mathrm{d}A_w = -\int_U (g_{n_k}) t_u \,\mathrm{d}A_w,$$

$$\int_U (g_{n_k})_v t \,\mathrm{d}A_w = -\int_U (g_{n_k}) t_v \,\mathrm{d}A_w.$$

Passing to the limit as $k \to \infty$, by the weak convergence of $(g_{n_k})_u$ and $(g_{n_k})_v$ and the uniform convergence of g_{n_k},

$$\int_U \hat{g} t \,\mathrm{d}A_w = -\int_U g t_u \,\mathrm{d}A_w \quad \text{and} \quad \int_U \hat{\hat{g}} t \,\mathrm{d}A_w = -\int_U g t_v \,\mathrm{d}A_w.$$

Thus, g has distributional derivatives in L^2, and so, by Lemma 2 of [9] or Theorem 2.1.4 of [272], g is ACL. (Note that, since g is continuous, it coincides with its representative in the sense used in Theorem 2.1.4 of [272].) Moreover, g_w is in $L^2(U)$ and thus in $L^2(B)$. □

A.1.8 *Proof of the Main Results*

Proof of Theorem A.1. As mentioned in the beginning of Sect. A.1.3, we can extend $\mu(z)$ to a measurable function in the whole plane which satisfies condition (A) for any bounded measurable set B. The existence of the homeomorphism $f(z)$ of the plane onto itself is proved in Sect. A.1.4. By the statements in Sect. A.1.5, $f(z)$ is ACL, and its partial derivatives f_z and $f_{\bar{z}}$ are in $L^q(K)$, K being any compact subset of the plane and any $q < 2$. Moreover, they are distributional derivatives. By the results in Sect. A.1.6, they satisfy the Beltrami equation. By restricting f to Δ, this completes the proof of Theorem A.1. □

Proof of Theorem A.2. Since $\hat{f}$ and g have partials locally in L^2, then $\hat{f}(g(w))$ is absolutely continuous in the sense of Tonelli and a.e.

$$\frac{\partial}{\partial \bar{w}} \hat{f}\big(g(w)\big) = \hat{f}_z\big(g(w)\big) g_{\bar{w}}(w) + \hat{f}_{\bar{z}}\big(g(w)\big) g_w(w).$$

Except for a set E of measure 0, $\hat{f}_{\bar{z}}(z) = \mu(z)\hat{f}_z(z)$, and except for the set $f(E)$, $\hat{f}_{\bar{z}}\big(g(w)\big) = \mu\big(g(w)\big)\hat{f}_z\big(g(w)\big)$. Therefore,

$$\hat{f}_z\big(g(w)\big) g_{\bar{w}}(w) + \hat{f}_{\bar{z}}\big(g(w)\big) g_w(w) = \hat{f}_z\big(g(w)\big)\left[g_{\bar{w}}(w) + \mu\big(g(w)\big) g_w(w)\right],$$

except for the set $f(E)$.

Now, we can show that $g_w = 0$ and $g_{\bar{w}} = 0$ a.e. on $f(E)$. Obviously, we are done if $f(E)$ is a set of measure 0. If we assume that $f(E)$ has a positive measure, then

$$0 = \int_E \frac{1}{1-|\mu(z)|^2} \mathrm{d}A = \int_{f(E)} \frac{J_w(w)}{1-\left|\mu\big(g(w)\big)\right|^2} \mathrm{d}A_w = \int_{f(E)} \left|g_w(w)\right|^2 \mathrm{d}A_w.$$

Here, J_w is the Jacobian of g, and the change of variables is justified by Lemma 2.1, Chap. III of [152]. Therefore, $g_w = 0$ a.e. on $f(E)$. Since g is sense preserving, $g_{\bar{w}} = 0$ a.e. on $f(E)$.

Now, let F and G be the sets where $f(z)$ and $g(z)$ are not differentiable, respectively. Let H be the set where f does not satisfy (β). By the same argument as above, it follows that, since F and H have measures 0, $g_w = 0$ and $g_{\bar{w}} = 0$ a.e. on $f(F) \cup f(H)$.

One can show, using differentiability a.e., that except on the set $f(F) \cup G$,

$$g_w(w) = \frac{f_z\big(g(w)\big)}{J_z\big(g(w)\big)} \quad \text{and} \quad g_{\bar{w}} = -\frac{f_{\bar{z}}\big(g(w)\big)}{J_z\big(g(w)\big)},$$

where J_z is the Jacobian of f. Using a substitution with the above expressions, except on the set $f(E) \cup f(F) \cup G$, we have

$$\hat{f}_z\big(g(w)\big)\left[g_{\bar{w}}(w) + \mu\big(g(w)\big)g_w(w)\right] = \hat{f}_z\big(g(w)\big)\left[-\frac{f_{\bar{z}}\big(g(w)\big)}{J_z\big(g(w)\big)} + \mu\big(g(w)\big)\frac{f_z\big(g(w)\big)}{J_z\big(g(w)\big)}\right],$$

and except on the set $f(H)$

$$-f_{\bar{z}}\big(g(w)\big) + \mu\big(g(w)\big)f_z\big(g(w)\big) = 0.$$

Since G has measure 0 and g_w and $g_{\bar{w}}$ are 0 a.e., on $f(E) \cup f(F) \cup f(H)$, we can conclude that

$$\hat{f}_z\big(g(w)\big)\left[g_{\bar{w}}(w) + \mu\big(g(w)\big)g_w(w)\right] = 0 \qquad \text{a.e.}$$

This proves that

$$\frac{\partial}{\partial \bar{w}} \hat{f}\big(g(w)\big) = 0 \qquad \text{a.e.}$$

By Corollary 1, p. 31 of [9], $\hat{f}\big(g(w)\big)$ is a conformal mapping of the plane onto itself. Thus,

$$\hat{f}\big(g(w)\big) = aw + b \quad \text{with constants } a \neq 0 \text{ and } b.$$

Therefore,

$$\hat{f}(z) = af(z) + b.$$

□

Proof of Theorem A.3. We extend $\mu(z)$ to the whole plane, construct the homeomorphisms $f(z)$ and $g(z)$, and restrict $g(z)$ to $f(\Delta)$. Then $\hat{f}\big(g(w)\big)$ defined on $f(\Delta)$ satisfies the condition

$$\frac{\partial}{\partial \bar{w}} \hat{f}\big(g(w)\big) = 0$$

a.e. in $f(\Delta)$. Thus, $\hat{f}\big(g(w)\big)$ is a conformal mapping of $f(\Delta)$ onto $\Theta = \hat{f}(\Delta)$, which we call $\xi(w)$. □

A.1.9 Examples and Comparisons

Example. Here, we are going to construct an example of a function $\mu(z)$ which satisfies David's condition (Δ) but does not satisfy Lehto's condition (Λ_1). To do so, we construct two sequences $\{I_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ of subsets of the open unit interval I_0 using the following procedure by induction: Denote by C_1 a Cantor set in I_0 of measure $t, 0 < t < 1$. Then $I_1 = I_0 \setminus C_1$ is a set of countably many open intervals. Assume that I_{n-1} is constructed and it is a union of countably many open intervals. Then C_n is defined as the union of Cantor sets on the open subintervals of I_{n-1} each with a measure t times the measure of the corresponding subinterval. Then we define I_n as $I_n = I_{n-1} \setminus C_n$; thus, I_n is a union of countably many open intervals.

From now on, depending on the context, we denote by $l(A)$ the linear measure or by $m(A)$ the area measure of a measurable set A. Thus, we have

$$\begin{array}{lll}
I_1 = I_0 \setminus C_1, & l(C_1) = t, & l(I_1) = 1-t, \\
I_2 = I_1 \setminus C_2, & l(C_2) = t(1-t), & l(I_2) = (1-t)^2, \\
\vdots & & \\
I_n = I_{n-1} \setminus C_n, & l(C_n) = t(1-t)^{n-1}, & l(I_n) = (1-t)^n.
\end{array}$$

We define also $E_n = I_n \times I_n$ and $G_n = C_n \times C_n$. Then $m(E_n) = (1-t)^{2n}$. Also $G_n \subset E_{n-1}$, $E_n = E_{n-1} \setminus G_n$ and $E_{n-1} \supset G_l$, for $l \geq n$. Denote $E_\infty = \bigcap_{n=1}^{\infty} E_n$ and let $\mu(z)$ be a measurable function defined as

$$\mu(z) = \begin{cases} 1 - e_n, & \text{on } G_{n+1} \\ 1, & \text{on } E_\infty \\ 0, & \text{everywhere else.} \end{cases}$$

Here, e_n is a decreasing sequence of positive numbers defined as

$$e_n = \frac{\alpha}{2n \ln(1-t)},$$

where α is a positive constant.

To show that $\mu(z)$ satisfies David's condition (Δ), we choose $\varepsilon > 0$ to be a sufficiently small number, and we define $A = \{z : |\mu(z)| > 1 - \varepsilon\}$. There exist numbers e_n and e_{n+1} such that $e_{n+1} \le \varepsilon < e_n$. Therefore, $A \subset \{z : |\mu(z)| > 1 - e_n\} \subset E_{n+1}$. Thus, $m(A) < m(E_{n+1}) = (1-t)^{2(n+1)} = \mathrm{e}^{-\alpha/e_{n+1}} \le \mathrm{e}^{-\alpha/\varepsilon}$. Therefore, $\mu(z)$ satisfies David's condition (Δ). However, $\mu(z) = 1$ on the set E_∞, which is not compact and whose closure is the closed unit square Q. Therefore, μ is not less than 1 outside of a compact set of measure 0 and thus does not satisfy Lehto's condition (Λ_1).

Comparisons. In this section, we relate (A.1.2) and (A.1.3) of Theorem A.1 to David's condition (A.1.1). We show that (A.1.1) implies (A.1.2) and (A.1.3) but (A.1.2) does not imply (A.1.1).

Lemma A.7. *Let K be a bounded measurable subset of the plane and μ be a complex-valued measurable function on K with $\|\mu\|_\infty \le 1$. Then the following two conditions are equivalent:*

(C) There exists $\beta > 0$ such that

$$\int_K \exp\left\{\frac{\beta}{1-|\mu|}\right\} \mathrm{d}A < \Phi,$$

where Φ is a positive constant.

(D) There exists $\alpha > 0$ and $C > 0$ such that, for sufficiently small $\varepsilon > 0$,

$$\text{measure}\{z : |\mu(z)| > 1 - \varepsilon\} \le C\mathrm{e}^{-\alpha/\varepsilon}.$$

Proof. We first show that (D) implies (C). Let

$$S_n = \left\{z : \frac{1}{2^{n+1}} \le 1 - |\mu| < \frac{1}{2^n}\right\}.$$

Therefore, measure $\{S_n\} \le C\mathrm{e}^{-\alpha 2^n}$ for $n > n_0$. Thus,

$$\begin{aligned}
\int_K \exp\left(\frac{\beta}{1-|\mu|}\right) \mathrm{d}A &= \int_{K \setminus \bigcup_{n>n_0} S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) \mathrm{d}A + \int_{\bigcup_{n>n_0} S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) \mathrm{d}A \\
&\le A_1 + \sum_{n=n_0}^{\infty} \int_{S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) \mathrm{d}A \\
&\le A_1 + \sum_{n=1}^{\infty} C \exp\{2\beta - \alpha\} 2^n,
\end{aligned}$$

where A_1 is an appropriate constant. If $\alpha > 2\beta$, this series converges, which proves that (D) implies (C). Now, we will show that (C) implies (D).

Let M be any positive number, $M > 1$. The set where $\exp\{\beta/(1-|\mu|)\} > M$ has measure less than Φ/M. Therefore, the set where $1-|\mu| < \beta/\log M$ has measure less than Φ/M. Choosing $\varepsilon = \beta/\log M$, the set where $|\mu| > 1-\varepsilon$ has measure less than $\Phi e^{-\beta/\varepsilon}$. This completes the proof that (C) implies (D). □

Lemma A.8. *Let $\mu(z)$ be defined in the plane. David's condition (A.1.1) implies (D) for every bounded measurable set K. However, condition (D) for every bounded measurable set K does not imply (A.1.1).*

Proof. The statement that (A.1.1) implies (D) for every bounded measurable set K is obvious. To prove the second part, let

$$\mu(z) = \mu(x+\mathrm{i}y) = \begin{cases} 1 - \dfrac{1}{\log(1/y)} & 0 < y < 1,\ x \in R, \\ 0 & \text{elsewhere.} \end{cases}$$

Then for some small $\varepsilon > 0$, the set where $1-|\mu| < \varepsilon$ is the set where $y \le \exp -1/\varepsilon$. Let K be a bounded measurable set with $|x| < l$ on K. The measure of the set $1-|\mu| < \varepsilon$ is less than $2l e^{-1/\varepsilon}$. Therefore, condition (D) is satisfied for any K. However, the measure of the set in the whole plane where $1-|\mu| < \varepsilon$ is infinite, and therefore, condition (A.1.1) does not hold. □

Lemma A.9. *Condition (D) for every bounded measurable set in the plane implies (A.1.2). Condition (A.1.2) does not imply condition (D) for every bounded measurable set K.*

Proof. To prove the first part of the statement, we use Lemma A.7 and the fact that (C) for every bounded measurable set implies (A.1.2).

To prove the second part of the statement, we give an example of a function μ for which (A.1.2) holds but (D) does not hold for some bounded measurable set.

Let $\{S_n\}$ be a set of disjoint disks in the plane contained in a finite disk S with

$$\text{measure}\{S_n\} = e^{-\frac{2^{n+1}}{n\log(1.5)}} - e^{-\frac{2^{n+2}}{(n+1)\log(1.5)}}.$$

Let $\mu(z) = 0$ in the complement of $\bigcup_{n=1}^{\infty} S_n$ and $\mu(z) = 1 - 1/2^{n+1}$ on S_n. Let K be any bounded measurable set. Then

$$\int_K \exp \frac{\dfrac{1}{1-|\mu|}}{1+\log\left(\dfrac{1}{1-|\mu|}\right)}\, \mathrm{d}A \le C + \sum_{n=1}^{\infty} \int_{S_n} \exp \frac{2^{n+1}}{1+(n+1)\log 2}\, \mathrm{d}A$$

$$\le C + \sum_{n=1}^{\infty} \exp\{-2^{(n+1)/2}\} < \infty.$$

Thus, (A.1.2) holds for μ.

Assume that condition (D) holds for S and the above constructed $\mu(z)$. Then the measure of the set $\{z : 1-|\mu| < 1/2^n\}$ is $\leq C\mathrm{e}^{-\alpha 2^n}$ for some constant α. On the other hand, the set $\{z : 1-|\mu| < 1/2^n\}$ has measure $\mathrm{e}^{(-2^{n+1})/(n\log(1.5))}$. Therefore,

$$\mathrm{e}^{\frac{-2^{n+1}}{n\log(1.5)}} \leq C\mathrm{e}^{-\alpha 2^n};$$

thus, $\mathrm{e}^{2^n(-\frac{2}{(n\log(1.5))}+\alpha)} \leq C$, as $n \to \infty$, which is a contradiction. Therefore, (C) does not hold for $\mu(z)$ on the set S. □

Lemma A.10. *If $\mu(z)$ satisfies condition (A.1.1), then*

$$\int_{|z|<R} \frac{1}{1-|\mu|}\,\mathrm{d}A = O(R^2).$$

Proof. Since (A.1.1) implies (D) for every bounded measurable set, we can choose a suitable $\varepsilon > 0$ such that (D) holds for $|z| < R$ and for some constants C and α. Using the same argument as in Lemma A.7, one can show that for every $R > 0$

$$\int_{\substack{1-|\mu|<\varepsilon \\ |z|<R}} \frac{1}{1-|\mu|}\,\mathrm{d}A < \Phi,$$

where Φ is a constant independent of R, while

$$\int_{\substack{1-|\mu|\geq\varepsilon \\ |z|<R}} \frac{1}{1-|\mu|}\,\mathrm{d}A \leq \frac{\pi R^2}{\varepsilon}.$$

This shows that

$$\int_{|z|<R} \frac{1}{1-|\mu|}\,\mathrm{d}A = O(R^2), \qquad \text{as } R \to \infty.$$

□

Lemma A.11. *No condition of the form*

$$\int_K \left(\frac{1}{1-|\mu|}\right)^{\lambda} \mathrm{d}A < \Phi$$

is sufficient for the conclusion of Proposition 1, where K is a bounded measurable set, Φ is a constant that depends on K and $\lambda \geq 1$.

Proof. We define

$$f(\rho\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\rho^{\tau}}\mathrm{e}^{\mathrm{i}\theta},$$

where $0<\tau<1$ and $0<\rho<1$. The complex dilatation of the mapping is

$$\mu(\rho e^{i\theta}) = e^{2i\theta}\left(\frac{\tau\rho^{\tau}-1}{\tau\rho^{\tau}+1}\right),$$

thus $1/(1-|\mu|) = 1/2+1/(2\tau\rho^{\tau})$. Therefore,

$$\iint\limits_{\rho<1} \frac{1}{(1-|\mu|)^{\lambda}} \mathrm{d}A = \iint\limits_{|\rho|<1} \left(\frac{1}{2}+\frac{1}{2\tau\rho^{\tau}}\right)^{\lambda} \mathrm{d}A.$$

This is finite if and only if

$$\int_0^1 \frac{1}{\rho^{\tau\lambda-1}} \mathrm{d}\rho < \infty.$$

Given λ, we can always choose $\tau < 2/\lambda$. But f maps the punctured disc $0<|z|<1$ onto a proper annulus $1<|z|<e$, and therefore, we do not get the conclusion of Proposition A.1 for this f. □

A.1.10 Formulations of New Existence Theorems

Here, we give results from the paper [57]. Let $h(x)$ be a convex, increasing function defined on $[1,\infty)$ such that $h(x) \geqslant C_{\lambda}x^{\lambda}$ for any $\lambda>1$ with $C_{\lambda}>0$. From now on, we will assume also that

$$\int\limits_1^{\infty} \frac{1}{th^{-1}(t)} \,\mathrm{d}t = \infty. \tag{A.1.12}$$

Theorem A.4. *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ, with $||\mu||_{\infty} \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\int\limits_B h\left(\frac{1}{1-|\mu|}\right) \mathrm{d}A < \Phi_B. \tag{A.1.13}$$

Then there exists an ACL *homeomorphism $f(z)$ of Δ into the plane which satisfies the Beltrami equation (B) a.e., with partials f_z and $f_{\bar{z}}$, locally in L^q, for $0<q<2$. The partials are also distributional derivatives. The inverse $g(\omega) = f^{-1}(\omega)$ is* ACL *in $f(\Delta)$ and has partials g_{ω} and $g_{\overline{\omega}}$ locally in L^2.*

Theorem A.5. *If Δ is the plane and if, in addition to (A.1.13), $\mu(z)$ satisfies*

$$\int_{\{|z|<R\}} \frac{1}{1-|\mu|}\,\mathrm{d}A = O\left(R^2\right), \quad R \to \infty,$$

then there exists an ACL *homeomorphism f which maps the plane onto itself with all the properties listed in Theorem A.4.*

A.1.11 Auxiliary Results and an Equivalent Statement

Let $h(x)$ be the function defined in Sect. A.1.10. Denote $\theta(x) = \ln(h(x))$ for x greater than some constant $c \geqslant 1$, such that $h(c) > e$; $\theta(x)$ is a positive increasing function in $[\ln h(c), \infty)$. Next, we show that the following conditions,

$$\int_{c_1}^{\infty} \frac{\mathrm{d}x}{xh^{-1}(x)} = \infty \tag{A.1.14}$$

and

$$\int_{c_2}^{\infty} \frac{\theta(x)}{x^2}\mathrm{d}x = \infty, \tag{A.1.15}$$

hold simultaneously, where c_1 and c_2 are suitable constants. The result can be stated as:

Lemma A.12. *Conditions (A.1.14) and (A.1.15) are equivalent.*

Proof. Make a change of variables in $\int_{c_1}^{\infty} \frac{\mathrm{d}x}{xh^{-1}(x)}$ by using the substitution $y = \ln(x)$. Then the last integral becomes $\int_{c^*}^{\infty} \frac{\mathrm{d}x}{\theta^{-1}(x)}$, where $c^* = \ln c_1$. Since $\frac{1}{\theta^{-1}(x)}$ and $\theta\left(\frac{1}{x}\right)$ are inverses of each other, it follows that $\int_{c^*}^{\infty} \frac{\mathrm{d}x}{\theta^{-1}(x)}$ is divergent if $\int_0^{c_*} \theta\left(\frac{1}{x}\right)\mathrm{d}x$ is for some suitable constant $c_* < 1$. After another substitution $y = \frac{1}{x}$, we obtain that the divergence of the last integral is equivalent to the divergence of $\int_{c_3}^{\infty} \frac{\theta(x)}{x^2}\mathrm{d}x$, where $c_3 = \frac{1}{c_*}$. □

A statement equivalent to Theorem A.4 follows from the auxiliary results above.

Theorem A.6. *Let Δ be a plane domain and $\mu(z)$ a measurable function defined a.e. in Δ, with $||\mu||_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\int_B \exp\left(\theta\left(\frac{1}{1-|\mu|}\right)\right)\mathrm{d}A < \Phi_B.$$

If

$$\int_1^\infty \frac{\theta(x)}{x^2}\,\mathrm{d}t = \infty,$$

there exists an ACL *homeomorphism* $f(z)$ *of* Δ *into the plane which satisfies the Beltrami equation (B) a.e., with partials* f_z *and* $f_{\overline{z}}$*, locally in* L^q*, for* $0 < q < 2$*. The partials are also distributional derivatives. The inverse* $g(\omega) = f^{-1}(\omega)$ *is* ACL *in* $f(\Delta)$ *and has partials* g_ω *and* $g_{\overline{\omega}}$ *locally in* L^2*.*

A.1.12 Construction of a Solution

Here, we assume that $\mu(z)$ satisfies condition (A.1.13), with $h(z)$ satisfying (A.1.12). In Δ, we define $\mu_n, n = 1,2,\ldots$, so that

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leqslant 1 - 1/n, \\ 0, & \text{if } |\mu(z)| > 1 - 1/n. \end{cases}$$

From the theory of quasiconformal mappings, we know that there exist qc mappings f_n, $n = 1,2,\ldots$, of Δ into the plane with complex dilatations μ_n, $n = 1,2,\ldots$.

Let z_0 be a fixed point in the plane. For $r_2 > r_1 > 0$, denote by A the circular ring $A = \{z : r_1 < |z - z_0| < r_2\}$ and by $M_n(r_1, r_2)$ the module of its image under f_n.

Proposition A.11. *For any point* z_0 *and circular ring* $A = \{r_1 < |z - z_0| < r_2\}$*, the module* $M_n(r_1, r_2)$ *of the image of* A *under* f_n *tends uniformly to* ∞ *as* $r_1 \to 0$*.*

Proof. The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation μ_n, where $\mu_n = \mu_n(z) = \mu_n(z_0 + r\mathrm{e}^{\mathrm{i}\theta})$, as follows (see [198]):

$$M_n(r_1, r_2) \geqslant \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{|1 - \mathrm{e}^{-2\mathrm{i}\theta}\mu_n|^2}{1 - |\mu_n|^2}\mathrm{d}\theta} \frac{\mathrm{d}r}{r}.$$

Using this, we obtain

$$M_n(r_1, r_2) \geqslant \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{1}{1 - |\mu|}\mathrm{d}\theta} \frac{\mathrm{d}r}{r}.$$

For any z_0 in a compact subset T of the plane containing the disc $|z - z_0| < r_2$,

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} h\left(\frac{1}{1 - |\mu|}\right)\,\mathrm{d}\theta\,\frac{\mathrm{d}r}{r} \leqslant C,$$

where C depends only on the compact subset T and the choice of r_2.

Now, we have

$$r^2\int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right)\mathrm{d}\theta < \frac{2C}{\log\frac{r_2}{r_1}}$$

on a set E of logarithmic measure $\frac{1}{2}\log\frac{r_2}{r_1}$. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right)\mathrm{d}\theta < \frac{C}{\pi r^2\log\frac{r_2}{r_1}} \quad \text{on } E.$$

Using the convexity of $h(x)$, we have

$$h\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}\,\mathrm{d}\theta\right) < \frac{C}{\pi r^2\log\frac{r_2}{r_1}} \quad \text{on } E$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}\,\mathrm{d}\theta < h^{-1}\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right) \quad \text{on } E.$$

From the estimates of the module and monotonicity properties of $h(x)$, we have

$$M_n(r_1,r_2) \geqslant \frac{1}{8\pi}\int_{r_1}^{r_2}\frac{1}{h^{-1}\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)}\frac{\mathrm{d}r}{r} \geqslant \frac{1}{8\pi}\int_{r_1}^{\sqrt{r_1r_2}}\frac{1}{h^{-1}\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)}\frac{\mathrm{d}r}{r}.$$

Now, we consider a monotonically decreasing sequence $\{s_k\}_{k=1}^{\infty}$ of positive numbers tending to 0 such that each interval $[s_{k+1},s_k]$ has the same logarithmic length, where $s_k/s_{k+1}=c$. By a ring decomposition, we mean a family of rings $r_1^{(j)}<|z-z_0|<r_2^{(j)}$ with $r_2^{(j+1)}\leqslant r_1^{(j)}$ and $r_1^{(j)}$ and $r_2^{(j)}\to 0$ as $j\to\infty$.

We take two ring decompositions with

$$r_1^{(j)} = s_{2j+1}, \quad r_2^{(j)} = s_{2j-1},$$
$$\hat{r}_1^{(j)} = s_{2j+2}, \quad \hat{r}_2^{(j)} = s_{2j}.$$

Now,

$$M_n\left(r_1^{(j)},r_2^{(j)}\right) \geqslant \frac{1}{8\pi}\sum_{j=1}^{\infty}\int_{s_{2j+1}}^{s_{2j}}\frac{1}{h^{-1}\left(\frac{C}{\pi r^2\log c}\right)}\frac{\mathrm{d}r}{r}$$

while

$$\sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi} \sum_{j=1}^{\infty} \int_{s_{2j+2}}^{s_{2j+1}} \frac{1}{h^{-1}\left(\frac{C}{\pi r^2 \log c}\right)} \frac{\mathrm{d}r}{r},$$

so

$$M_n\left(r_1^{(j)}, r_2^{(j)}\right) + \sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi} \int_{0}^{s_1} \frac{1}{h^{-1}\left(\frac{C}{\pi r^2 \log c}\right)} \frac{\mathrm{d}r}{r}.$$

With a change of variables $t = C/\pi r^2 \log c$, this last term becomes equal to $\frac{1}{8\pi}\int_{\bigstar}^{\infty} \frac{1}{th^{-1}(t)}\mathrm{d}t$, which is a well-defined lower limit. Thus, at least one of the ring decompositions has module sum bounded below by $\frac{1}{16\pi}\int_{\bigstar}^{\infty} \frac{1}{th^{-1}(t)}\mathrm{d}t$ and therefore approaches ∞ uniformly with respect to n and z_0. From the superadditivity property of the module, it follows that $\lim_{r_1 \to 0} M_n(r_1, r_2) = \infty$, uniformly with respect to z_0 and n. □

From now on, we shall assume that the quasiconformal mappings $\{f_n(z)\}$, $n = 1, 2 \ldots$ have two fixed points a_1 and a_2, with $d = |a_2 - a_1|$.

The next proposition follows from Proposition A.2 and Arzela–Ascoli theorem.

Proposition A.12. *For the sequence $\{f_n(z)\}$, there exists a subsequence of functions which converges uniformly to a function $f(z)$ on compact subsets.*

Here, we follow the proof of Theorem A.4 above.

A.1.13 The Solution Is a Homeomorphism

In the same manner as above in Sect. A.1.4, one can prove:

Proposition A.13. *The function $f(z)$ constructed in Proposition A.12 is a homeomorphism of Δ into the plane.*

A.1.14 Differentiability Properties of the Solution

In the same manner as in the proofs of Lemmas A.3 and A.4, respectively, one can prove:

Proposition A.14. *The function $f(z)$ is* ACL.

Proposition A.15. *The partials f_z and $f_{\bar{z}}$ are in L^q on compact subsets of Δ for every $q < 2$.*

Thus, $f(z)$ has generalized L^q-derivatives according to the terminology introduced in [152].

Using the same methods as in the proof of Lemmas A.5, one can prove:

Proposition A.16. *The function $f(z)$ satisfies the Beltrami equation.*

A.1.15 The Inverse Mapping

In the same manner as in the proof of Proposition A.10, one can prove

Proposition A.17. *The function g is* ACL*, and g_w and $g_{\overline{w}}$ are locally in L^2.*

So far, we have proved Theorems A.4 and A.5.

A.1.16 The Case of the Whole Plane

In the same manner as in the proof of Proposition A.4, one can prove

Proposition A.18. *If*

$$\int\limits_{|z|<R} \frac{1}{1-|\mu|}\, \mathrm{d}A = O\left(R^2\right) \qquad \text{as } R \to \infty,$$

then $f_n(z)$ converges uniformly to ∞ as $z \to \infty$.

This proposition and the rest of the results imply Theorem A.4. This concludes the proofs of Theorems A.4 and A.5.

A.2 The Existence Theorems of Gutlyanski–Martio–Sugawa–Vuorinen

A.2.1 Introduction

The analytic theory of plane quasiconformal mappings f is based on the Beltrami partial differential equation

$$f_{\bar{z}} = \mu(z)\, f_z \quad \text{a.e.} \tag{A.2.1}$$

with the complex-valued measurable coefficient μ satisfying the uniform ellipticity assumption $\|\mu\|_\infty < 1$. In the case $|\mu(z)| < 1$ a.e. in $\mathbb{C}$ and $\|\mu\|_\infty = 1$, (A.2.1) is called a degenerate Beltrami equation, and the structure of the solutions depends heavily on the degeneration of μ. In this article, unless otherwise stated, by a *Beltrami coefficient* in a domain Ω, we mean a complex-valued measurable function

μ in Ω such that $|\mu| < 1$ a.e. in Ω, and by a *solution* to the Beltrami equation (A.2.1) in a domain Ω, we mean a function f in the Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$ whose partial derivatives satisfy (A.2.1) in Ω. Then f is called μ*-conformal* in Ω. The measurable Riemann mapping theorem (cf. [9]) states that, given a measurable function μ in the plane $\mathbb{C}$ with $\|\mu\|_\infty < 1$, there is a quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$, $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$, satisfying (A.2.1). Contrary to this, the degenerate Beltrami equation need not have a homeomorphic solution and a solution, if it exists, need not be unique. See, for instance, [116]. Therefore, in order to obtain existence or uniqueness results, some extra constraints must be imposed on μ.

The degeneration of μ is usually expressed in terms of the pointwise maximal dilatation function

$$K(z) = K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}. \tag{A.2.2}$$

This takes into account the absolute value of μ only. We show that for sharper results, the argument of $\mu(z)$ should also be considered. For example, consider the Beltrami coefficients $\mu_1(z) = (1-|z|)z/\bar{z}$ and $\mu_2(z) = (|z|-1)z/\bar{z}$ defined in the unit disk $\mathbb{D}$. It is immediate that $\|\mu_j\|_\infty = 1$, $j = 1,2$, and $K_{\mu_1}(z) = K_{\mu_2}(z)$ whenever $z \in \mathbb{D} \setminus \{0\}$. The radial stretching $f_1 : \mathbb{D} \to \mathbb{D}$, defined by

$$f_1(z) = \frac{z}{|z|^2} \mathrm{e}^{2(1-1/|z|)} \tag{A.2.3}$$

for $z \in \mathbb{D} \setminus \{0\}$, $f_1(0) = 0$, is μ_1-conformal. The second radial stretching

$$f_2(z) = \frac{z}{|z|(2-|z|)} \tag{A.2.4}$$

in the punctured disk $\mathbb{D} \setminus \{0\}$ is μ_2-conformal and has the cavitation effect since it maps $\mathbb{D} \setminus \{0\}$ homeomorphically onto the annulus $1/2 < |z| < 1$. Actually, the continuous solution to the Beltrami equation with $\mu = \mu_2$ is unique up to the post-composition of analytic functions (cf. Proposition A.22 below), and hence, the cavitation is inevitable in this case. Thus, the cavitation problem requires more precise information on μ than merely on $|\mu|$.

To study the aforementioned problem, we employ the angular dilatation coefficient D_{μ,z_0} (see (A.2.6) below) to take into account an effect of the argument of μ as well. On one hand, it allows us to prove the existence of a homeomorphic solution f to the Beltrami equation (A.2.1) for a given Beltrami coefficient μ with $\|\mu\|_\infty = 1$ provided that D_{μ,z_0} satisfies a local integrability condition for each z_0; see Theorem A.10. We also obtain an estimate for the modulus of continuity of f. On the other hand, we establish a uniqueness theorem for the solution of (A.2.1) in the case when the singular sets of μ are totally disconnected compacta with certain geometric conditions involving D_{μ,z_0}; see Theorem A.12. The modulus estimate for annuli in Lemma A.18 in terms of the integral means of the angular dilatation coefficients plays a crucial role in the proof of the existence and uniqueness results. A normal family argument is also used; see Propositions A.19 and A.20.

The idea of employing μ instead of $|\mu|$ in the study of some regularity problems for quasiconformal mappings is due to Andreian Cazacu [15] and Reich and Walczak [198]. Lehto [148, 149] was the first who considered the degenerate Beltrami equation from this point of view.

The degeneration of μ in terms of $|\mu(z)|$ or $K_\mu(z)$ has recently been extensively studied. This is due to the close connection of f in (A.2.1) to the solutions of elliptic partial differential equations. For the earlier studies of μ-homeomorphisms, we refer to [30, 34, 174, 185]. The results of Pesin [185] have been substantially extended by Brakalova and Jenkins [56]. For the recent deep theorems on the existence and uniqueness of μ-homeomorphisms, see Iwaniec and Martin [116], who extended the well-known results of David [70] and Tukia [254], and see also [65, 211, 234] and the references therein.

Let us indicate a couple of features of our main results. First, by virtue of adoption of the angular dilatation coefficients, the existence theorem and its primitive, Theorem A.9, cover many cases when K_μ fails to satisfy known integrability conditions; see, for instance, Examples A.1 and A.2. Even the case when the singular set of μ consists of finitely many points is of independent interest; see Sect. 10.2.3.

Theorem A.7. *Let $\mu(z)$ be a Beltrami coefficient in $\widehat{\mathbb{C}}$ such that the set of singularity* $\mathrm{Sing}(\mu)$ (*see* Remark A.1) *consists of finitely many points. Then there exists a homeomorphism $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ which is locally quasiconformal in $\widehat{\mathbb{C}}\setminus\mathrm{Sing}(\mu)$ and whose complex dilatation μ_f satisfies $|\mu_f(z)|=|\mu(z)|$ a.e.*

Secondly, Theorem A.10 applied to the classical setting involving K_μ only gives the following result:

Theorem A.8. *Suppose that a Beltrami coefficient μ on $\mathbb{C}$ satisfies*

$$\iint_{\mathbb{C}} \mathrm{e}^{H(K_\mu(z))}\frac{\mathrm{d}m(z)}{(1+|z|^2)^2}<+\infty$$

for a measurable function $H:[1,+\infty)\to\mathbb{R}$ for which there exist an integer $n\geq 1$ and numbers $c>0,\ \alpha\in(-\infty,1]$ such that

$$H(t)\geq ct/(\log t)(\log_2 t)\cdots(\log_{n-1}t)(\log_n t)^{\alpha}\quad \text{for large } t.$$

Then there exists a homeomorphic solution $f:\mathbb{C}\to\mathbb{C}$ to the Beltrami equation with μ such that $f\in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ for every $1\leq q<2$ and $f^{-1}\in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Moreover, for $\alpha<1, n\geq 2$ and for $\alpha\in[0,1)$ if $n=1$, f satisfies the inequality

$$|f(z)-f(z_0)|\leq C\exp\left(\frac{-c}{2(1-\alpha)}\left(\log_{n+1}\frac{1}{|z-z_0|}\right)^{1-\alpha}\right),\quad |z-z_0|<\delta_0,$$

where the constants $C>0$ and $\delta_0>0$ are locally uniformly bounded above and below, respectively. If $n=1$ and $\alpha<0$, the above inequality still holds with the constant $c/(2(1-\alpha))$ being replaced by any larger number.

Here, $\log_n$ denotes the iterated logarithm; see Sect. A.2.5. Examples show that this result is close to being optimal. Indeed, for each n, one cannot take $\alpha > 1$ in the integrability condition above. Moreover, the constant $c/(2(1-\alpha))$ and the exponent $1-\alpha$ in the above estimate for modulus of continuity cannot be replaced by any smaller constants.

A.2.2 Sequences of Self-Homeomorphisms

The degenerate Beltrami equation need not have a homeomorphic solution nor even a nonconstant solution. A usual approach for the existence of a solution is to consider a sequence f_n of quasiconformal homeomorphisms satisfying (A.2.1) with the Beltrami coefficients μ_n, $\|\mu_n\|_\infty < 1$, such that $\mu_n \to \mu$ a.e., and then to use a normal family argument to obtain a limit mapping f. Some conditions must be imposed on μ in order to pick a converging subsequence.

In the following, we introduce a modulus method to study normal families of homeomorphisms. This will be employed to solve the degenerated Beltrami equation; however, the method is of independent interest.

We introduce some notation. We denote the Euclidean distance and the spherical distance between z and w by

$$d(z,w) = |z-w| \quad \text{and} \quad d^\sharp(z,w) = |z-w|/\sqrt{(1+|z|^2)(1+|w|^2)},$$

respectively. Also, we denote by $A(z_0,r,R)$ and by $A^\sharp(z_0,r,R)$ the (circular) annuli in the Euclidean and the spherical metric, respectively, i.e.,

$$A(z_0,r,R) = B(z_0,R) \setminus \overline{B}(z_0,r) \quad \text{and} \quad A^\sharp(z_0,r,R) = B^\sharp(z_0,R) \setminus \overline{B}^\sharp(z_0,r)$$

for $z_0 \in \widehat{\mathbb{C}}$ and $0 \le r < R$, where $B(z_0,r) = \{z \in \mathbb{C} : |z-z_0| < r\}$ and $B^\sharp(z_0,r) = \{z \in \widehat{\mathbb{C}} : d^\sharp(z,z_0) < r\}$. Here, in the case when $z_0 = \infty$, we set $B(\infty,r) = \{z \in \widehat{\mathbb{C}} : |z| > 1/r\}$, and hence, $A(\infty,r,R) = A(0,1/R,1/r)$. In the sequel, for subsets E, E_0, E_1 of $\widehat{\mathbb{C}}$, $\operatorname{diam} E$ and $\operatorname{dist}(E_0,E_1)$ stand for the diameter of E and the distance between E_0 and E_1, respectively, measured in the Euclidean metric d. Similarly, $\operatorname{diam}^\sharp E$ and $\operatorname{dist}^\sharp(E_0,E_1)$ stand for those measured in the spherical metric $d^\sharp$. We also denote by $\mathscr{A}$ and $\mathscr{A}^\sharp$ the two-dimensional Lebesgue measure and the spherical measure, respectively, i.e., $\mathscr{A}(E) = \iint_E \mathrm{d}m(z)$ and $\mathscr{A}^\sharp(E) = \iint_E (1+|z|^2)^{-2}\mathrm{d}m(z)$.

A doubly connected domain is called a ring domain. The modulus m of a ring domain A is the number such that A is conformally equivalent to $\{1 < |z| < \mathrm{e}^m\}$ and will be denoted by $\operatorname{mod} A$. When A is conformally equivalent to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we define $\operatorname{mod} A = \infty$. A nonnegative function $\rho(z,r,R)$ in $(z,r,R) \in \widehat{\mathbb{C}} \times (0,+\infty) \times (0,+\infty)$, $r < R$, will be called a *modulus constraint* if $\rho(z_0,r,R) \to +\infty$ as $r \to 0$ for any fixed $R \in (0,+\infty)$ and $z_0 \in \widehat{\mathbb{C}}$.

We denote by $\mathscr{H}_\rho$ the family of all normalized homeomorphisms $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that the condition

$$\operatorname{mod} f(A(z_0,r,R)) \geq \rho(z_0,r,R)$$

holds for all $z_0\in\widehat{\mathbb{C}}$ and $r,R\in(0,+\infty)$ with $r<R$. Here and below, a homeomorphism $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is said to be normalized if f fixes $0,1$, and ∞. Note that the above condition is not Möbius invariant.

Note also that, without changing the family $\mathscr{H}_\rho$, we may always assume the monotonicity condition $\rho(z_0,r_1,R_1)\leq\rho(z_0,r_2,R_2)$ for $z_0\in\widehat{\mathbb{C}}$ and $r_2\leq r_1<R_1\leq R_2$.

Similarly, a nonnegative function $\rho^\sharp(z,r,R)$ in $(z,r,R)\in\widehat{\mathbb{C}}\times(0,1)\times(0,1)$, $r<R$, will be called a *spherical modulus constraint* if $\rho^\sharp(z,r,R)\to+\infty$ as $r\to 0$ for any fixed $R\in(0,1)$ and $z\in\widehat{\mathbb{C}}$. We let $\mathscr{H}^\sharp_{\rho^\sharp}$ be the family of all normalized self-homeomorphisms f of $\widehat{\mathbb{C}}$ such that $\operatorname{mod} f(A^\sharp(z_0,r,R))\geq\rho^\sharp(z_0,r,R)$ holds for all $0<r<R<1$ and $z_0\in\widehat{\mathbb{C}}$.

A family of continuous maps from a domain Ω into $\widehat{\mathbb{C}}$ is said to be *normal* or *compact* if it is relatively compact or compact, respectively, with respect to the topology of local uniform convergence.

The following propositions are similar to a result in [149] where stronger assumptions were used (see also Lemma 1 in Sect. 4 of [56]):

Proposition A.19. *Let $\rho^\sharp$ be a spherical modulus constraint. Then*

1) *$\mathscr{H}^\sharp_{\rho^\sharp}$ is a compact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$.*
2) *Every $f\in\mathscr{H}^\sharp_{\rho^\sharp}$ satisfies the inequality*

$$d^\sharp(f(z_1),f(z_2))\leq C\mathrm{e}^{-\frac{1}{2}\rho^\sharp(z_0,r_1,r_2)},\quad z_1,z_2\in B^\sharp(z_0,r_1), \tag{A.2.1}$$

for $z_0\in\widehat{\mathbb{C}}$ and $0<r_1<r_2<1/2\sqrt{2}$, where C is an absolute constant.

Proposition A.20. *Let ρ be a modulus constraint. Then*

1) *$\mathscr{H}_\rho$ is a compact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$.*
2) *For each $R>0$, there is a constant $C=C(R,\rho)>0$ depending only on R and ρ such that every $f\in\mathscr{H}_\rho$ satisfies*

$$|f(z_1)-f(z_2)|\leq C\mathrm{e}^{-\rho(z_0,r_1,r_2)},\quad z_1,z_2\in B(z_0,r_1), \tag{A.2.2}$$

for $|z_0|\leq R$ and $0<r_1<r_2<R$.

In order to prove the propositions, we prepare a few lemmas. For the proof of the first one, see [106], where the authors assert that one can take $C_0=\pi^{-1}\log 2(1+\sqrt{2})=0.50118\ldots$ below. In fact, it essentially follows from the famous Teichmüller lemma on his extremal ring domain.

Lemma A.13. *There exists a universal constant $C_0 > 0$ with the property that for a ring domain B in $\mathbb{C}$ with $\operatorname{mod} B > C_0$ which separates a point z_0 from ∞, we can choose an annulus A in B of the form $A = A(z_0, r_1, r_2)$, $r_1 < r_2$, so that* $\operatorname{mod} A \geq \operatorname{mod} B - C_0$.

We need information about the size of components of the complement of a ring domain of sufficiently large modulus. There are several such estimates, and the following form due to Lehto–Virtanen [152, Lemma I.6.1] is probably the best known. Let B be a ring domain whose complement in $\widehat{\mathbb{C}}$ consists of continua E_0 and E_1. Then

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \leq \frac{\pi}{\sqrt{2 \operatorname{mod} B}}.$$

However, this bound can be improved when $\operatorname{mod} B$ is large enough. The following result provides an estimate which has a sharp coefficient of $\operatorname{mod} B$ in the exponent:

Lemma A.14. *Let B be an arbitrary ring domain in $\widehat{\mathbb{C}}$ and let E_0 and E_1 be the components of $\widehat{\mathbb{C}} \setminus B$. Then the inequality*

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \leq C_1 \mathrm{e}^{-\frac{1}{2}\operatorname{mod} B}$$

holds, where C_1 is an absolute constant.

Proof. We may assume that $\infty \in E_1$. Then we get the desired conclusion by combining Lemma A.13 with the following elementary but sharp result. In particular, we can use the value $C_1 = 2\mathrm{e}^{C_0/2} = 2.56957\ldots$, where C_0 is the constant in Lemma A.13. □

Lemma A.15. *Let A be an annulus in $\widehat{\mathbb{C}}$ whose complement consists of disjoint closed disks E_0 and E_1. Then*

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \leq \frac{1}{\cosh(\frac{1}{2}\operatorname{mod} A)}.$$

Equality holds if and only if $\operatorname{diam}^\sharp E_0 = \operatorname{diam}^\sharp E_1$ *and if the spherical centers of E_0 and E_1 are antipodal.*

Proof. If $\operatorname{diam}^\sharp E_j > \operatorname{diam}^\sharp E_{1-j}$ for some $j = 0, 1$, then we can decrease $\operatorname{diam}^\sharp E_j$ while leaving $\operatorname{diam}^\sharp E_{1-j}$ and $\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\}$ invariant, and the resulting annulus will have larger modulus. Hence, we may assume $\operatorname{diam}^\sharp E_0 = \operatorname{diam}^\sharp E_1$. Performing a suitable isometric Möbius transformation with respect to the spherical metric, we can further assume that E_0 and E_1 are symmetric in the imaginary axis and that the center of E_0 is a positive real number. Let $E_0 \cap \mathbb{R} = [r, R]$ and δ be the hyperbolic diameter of E_0 in the hyperbolic plane $\mathbb{H} = \{z : \operatorname{Re} z > 0\}$. Note that

$$\delta = \int_r^R \frac{\mathrm{d}x}{2x} = \frac{1}{2}\log\frac{R}{r} = \log t,$$

where we set $t=\sqrt{R/r}=\mathrm{e}^{\delta}>1$. Since $\mathbb{H}\setminus E_0$ is Möbius equivalent to the annulus $A(0,\tanh(\delta/2),1)$, we can compute the modulus of A as follows:

$$\operatorname{mod} A = 2\operatorname{mod}(\mathbb{H}\setminus E_0) = 2\log\coth\frac{\delta}{2} = 2\log\frac{t+1}{t-1}.$$

In particular, we have $t=\coth(m/2)$ where $m=\operatorname{mod}A/2$. On the other hand,

$$\begin{aligned}\operatorname{diam}^{\sharp} E_0 = d^{\sharp}(r,R) &= \frac{R-r}{\sqrt{1+R^2}\sqrt{1+r^2}} = \frac{(t^2-1)r}{\sqrt{1+t^4r^2}\sqrt{1+r^2}}\\ &= \frac{t^2-1}{\sqrt{1+t^4+r^{-2}+t^4r^2}} \le \frac{t^2-1}{\sqrt{1+t^4+2t^2}} = \frac{t^2-1}{t^2+1}\\ &= \frac{\coth^2(m/2)-1}{\coth^2(m/2)+1} = \frac{1}{\cosh m},\end{aligned}$$

where equality holds if and only if $rt=1$ or, equivalently, $rR=1$. The last relation means that the spherical center of E_0 is 1 and vice versa. In this case, the spherical center of E_1 is -1, which is the antipode of 1. Hence, the last assertion of the lemma follows. □

Lemma A.14 has the sharp coefficient $1/2$ in the exponent; however, in the case of ring domains in the finite plane $\mathbb{C}$, the coefficient is no longer best possible. The following estimate has the sharp coefficient 1 in the exponent in this case:

Lemma A.16. *Let B be a ring domain in $\mathbb{C}$ whose complement in $\widehat{\mathbb{C}}$ consists of the bounded component E_0 and the unbounded component E_1. Then the inequality*

$$\operatorname{diam} E_0 \le C_3 \operatorname{dist}(E_0,E_1)\,\mathrm{e}^{-\operatorname{mod} B}$$

holds provided that $\operatorname{mod} B > C_2$, *where C_2 and C_3 are positive absolute constants.*

Proof. We may assume that $\operatorname{dist}(E_0,E_1)=1,\ 0\in E_0$ and $1\in E_1$. Let $a\neq 0$ be an arbitrary point in E_0. Then, by Teichmüller's modulus theorem (see [152]), we have

$$\operatorname{mod} B \le 2\mu\left(\sqrt{\frac{|a|}{1+|a|}}\right),$$

where $\mu(r)$ denotes the modulus of the Grötzsch ring $B(0,1)\setminus[0,r]$. Using the well-known estimate $\mu(r)<\log(4/r)$, we obtain

$$|a| \le \frac{16}{\mathrm{e}^{\operatorname{mod} B}-16} \le 32\,\mathrm{e}^{-\operatorname{mod} B}$$

if $\operatorname{mod} B > 5\log 2$. Hence, $\operatorname{diam} E_0 \le 64\,\mathrm{e}^{-\operatorname{mod} B}$ whenever $\operatorname{mod} B > 5\log 2$. In particular, the assertion holds for $C_2 = 5\log 2$ and $C_3=64$. □

Proof of Proposition A.19. Observe first that $d^\sharp(0,1) = d^\sharp(1,\infty) = 1/\sqrt{2}$ and that $d^\sharp(0,\infty) = 1$. In particular, for $r_2 < 1/2\sqrt{2}$, the disk $B = B^\sharp(z_0,r_2)$ cannot contain more than one of the three fixed points $0,1,\infty$. Therefore, the component $E_1 = \widehat{\mathbb{C}} \setminus f(B)$ of the complement of $f(A^\sharp(z_0,r_1,r_2))$ has spherical diameter at least $1/\sqrt{2}$. Meanwhile, the other component $E_0 = f(\overline{B}^\sharp(z_0,r_1))$ has spherical diameter at most 1. Consequently, $\operatorname{diam}^\sharp E_0 \leq \sqrt{2}\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\}$. Now, inequality (A.2.1) follows from Lemma A.14 with $C = \sqrt{2}C_1$, where C_1 is the constant appearing in the lemma. This inequality in turn implies the equicontinuity of $\mathscr{H}^\sharp_{\rho^\sharp}$. Since $\widehat{\mathbb{C}}$ is compact, then by the Arzelà–Ascoli theorem, the family $\mathscr{H}^\sharp_{\rho^\sharp}$ is normal.

Let f be the uniform limit of a sequence f_n in $\mathscr{H}^\sharp_{\rho^\sharp}$. We show that f is a member of $\mathscr{H}^\sharp_{\rho^\sharp}$. Since the mapping degree is preserved under uniform convergence, f has degree 1, in particular, $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a surjective continuous map. We now consider the open set

$$V = \{z \in \widehat{\mathbb{C}}; f \text{ is locally constant at } z\}.$$

First, we show the following:

Claim. If $z_0 \in \widehat{\mathbb{C}} \setminus V$, then $f(z) \neq f(z_0)$ for $z \in \widehat{\mathbb{C}} \setminus \{z_0\}$.

By permuting the roles of $0,1,\infty$ if necessary, we may assume that $z_0 \neq \infty$ and that $f(z_0) \neq \infty$. Pick a point $w_0 \neq z_0$. We will show that $f(w_0) \neq f(z_0)$. Choose a small positive number R so that $R < \min\{d^\sharp(z_0,w_0), d^\sharp(z_0,\infty)\}$. Then, by the definition of $\mathscr{H}^\sharp_{\rho^\sharp}$, $\inf_n \operatorname{mod}(f_n(A^\sharp(z_0,\delta,R))) > C_0$ for sufficiently small $\delta > 0$, where C_0 is the constant in Lemma A.13. By virtue of Lemma A.13, we can find an annulus A_n of the form $A(f_n(z_0), r_n, r'_n)$, $r_n < r'_n$, in $f_n(A^\sharp(z_0,\delta,R))$ for n large enough.

Since f is not locally constant at z_0, there exists a point z_1 in the disk $d^\sharp(z,z_0) < \delta$ with $f(z_0) \neq f(z_1)$. The annulus A_n separates $f_n(z_0), f_n(z_1)$ from $f_n(w_0), \infty$, so we obtain $|f_n(z_1) - f_n(z_0)| \leq r_n$ and $r'_n \leq |f_n(w_0) - f_n(z_0)|$. In particular, $|f_n(z_1) - f_n(z_0)| \leq |f_n(w_0) - f_n(z_0)|$ for n large enough. Letting $n \to \infty$, we obtain $0 < |f(z_1) - f(z_0)| \leq |f(w_0) - f(z_0)|$, and hence, $f(w_0) \neq f(z_0)$.

We next show that V is empty. Suppose that V has a nonempty component V_0. Then f takes a constant value, say b, in V_0. Because $V_0 \neq \widehat{\mathbb{C}}$, there is a point z_0 in ∂V_0. By continuity, we have $f(z_0) = b$. On the other hand, from the above claim, it follows that $f(z) \neq f(z_0) = b$ for any point z other than z_0, which contradicts the fact that $f = b$ in V_0. Thus, we conclude that V is empty, namely, f is not locally constant at any point.

By using the claim again, we obtain $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. Thus, the injectivity of f follows. Finally, the continuity of moduli of ring domains with respect to the Hausdorff convergence (see [87]) implies that $f \in \mathscr{H}^\sharp_{\rho^\sharp}$.

Proof of Proposition A.20. It is easy to see that there is a spherical modulus constraint $\rho^\sharp$ such that $\mathscr{H}_\rho \subset \mathscr{H}^\sharp_{\rho^\sharp}$. From Proposition A.19, the compactness of $\mathscr{H}_\rho$ now follows. We next give an estimate of $\operatorname{dist}(E_0,E_1)$, where E_0 and E_1 are the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus f(A)$, $A = A(z_0, r_1, r_2)$, respectively. Since $\mathscr{H}_\rho$ is compact, $S = \sup\{|g(z)| : g \in \mathscr{H}_\rho, |z| \le 2R\} < +\infty$. Now, it is clear that $\mathrm{dist}(E_0, E_1) \le \mathrm{diam}\, f(\partial B(z_0, r_2)) \le 2S$. Set $m = \mathrm{mod}\, f(A)$. If $m > C_2$, we obtain $\mathrm{diam}\, E_0 \le 2SC_3 \mathrm{e}^{-m}$ by Lemma A.16. Otherwise, we have $\mathrm{diam}\, E_0 \le 2S \le 2S\mathrm{e}^{C_2}\mathrm{e}^{-m}$. Hence, the proof is complete. □

Let μ be a Beltrami coefficient defined in $\mathbb{C}$ with $\|\mu\|_\infty \le 1$. In order to apply Proposition A.20 to the study of the Beltrami equation, we first introduce a standard approximation procedure. For $n = 1, 2, \ldots$, we set

$$\mu_n(z) = \mu(z), \quad \text{if } |\mu(z)| \le 1 - 1/n, \tag{A.2.3}$$

and $\mu_n(z) = 0$ otherwise and denote by f_n the sequence of quasiconformal automorphisms of the extended complex plane preserving 0, 1 and ∞, and having μ_n as its complex dilatation. The existence of such f_n is guaranteed by the measurable Riemann mapping theorem. We will call such f_n the *canonical approximating sequence* for μ. The topological structure of the family $\{f_n\}$ with respect to the uniform convergence under some additional assumptions on μ will be described in Theorem A.9.

A function $H : [0, +\infty) \to \mathbb{R}$ is called a *dominating factor* if the following conditions are satisfied:

1. $H(x)$ is continuous and strictly increasing in $[x_0, +\infty)$ and $H(x) = H(x_0)$ for $x \in [0, x_0]$ for some $x_0 \ge 0$.
2. The function $\mathrm{e}^{H(x)}$ is convex in $x \in [0, +\infty)$.

The convexity of e^H implies that $H(x) \to +\infty$ as $x \to +\infty$. In the sequel, the inverse H^{-1} of H will mean the inverse of the homeomorphism $H : [x_0, +\infty) \to [H(0), +\infty)$.

A dominating factor H is said to be of *divergence type* if

$$\int_1^{+\infty} \frac{H(x)\mathrm{d}x}{x^2} = +\infty, \tag{A.2.4}$$

and otherwise H is said to be of *convergence type*. Denote by $\mathscr{D}$ the set of all dominating factors of divergence type. Note that $H(\eta x)$ and $\eta H(x)$ are dominating factors of the same type as $H(x)$ if $H(x)$ is a dominating factor and if η is a positive constant.

Lemma A.17. *Let H be a dominating factor. Then H is of divergence type if and only if*

$$\int_{t_1}^{+\infty} \frac{\mathrm{d}t}{H^{-1}(t)} = +\infty \tag{A.2.5}$$

for a sufficiently large number t_1.

Proof. By the change of variables $t = H(x)$ and integration by parts, we see that

$$\int_{t_1}^{t_2} \frac{\mathrm{d}t}{H^{-1}(t)} = \int_{x_1}^{x_2} \frac{\mathrm{d}H(x)}{x} = \frac{H(x_2)}{x_2} - \frac{H(x_1)}{x_1} + \int_{x_1}^{x_2} \frac{H(x)\mathrm{d}x}{x^2},$$

where $t_j = H(x_j)$ for $j = 1,2$. Now, the "only if" part follows because $H(x)/x$ is positive for large x. In order to show the "if" part, assume that (A.2.4) fails. Then, in view of the above identity, (A.2.5) would imply that $H(x)/x \to \infty$ if $x \to \infty$. In particular, $H(x)/x > C > 0$ holds for sufficiently large x where C is a positive constant. Hence, for x_1 large enough, we obtain $\int_{x_1}^{x_2} H(x)\mathrm{d}x/x^2 > \int_{x_1}^{x_2} C\,\mathrm{d}x/x = C\log(x_2/x_1) \to \infty$ as $x_2 \to \infty$, a contradiction. Thus, (A.2.4) follows from (A.2.5).

□

Let $\mu \in L^\infty(\Omega)$ be a Beltrami coefficient with $|\mu| < 1$ a.e. in an open subset Ω of $\widehat{\mathbb{C}}$. We define the *angular dilatation* D_{μ,z_0} of μ at $z_0 \in \widehat{\mathbb{C}}$ as

$$D_{\mu,z_0}(z) = \frac{\left|1 - \mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}\right|^2}{1 - |\mu(z)|^2} \tag{A.2.6}$$

in the case when z_0 is finite and as $D_{\mu,\infty}(z) = D_{\mu,0}(z)$ in the case when $z_0 = \infty$. Then D_{μ,z_0} is a measurable function in Ω and satisfies the inequality $1/K_\mu(z) \le D_{\mu,z_0}(z) \le K_\mu(z)$ a.e. for each $z_0 \in \widehat{\mathbb{C}}$. Note that $D_{\mu,0}(z) = K_\mu(z)$ holds a.e. if and only if $\mu(z)$ has the form $-\rho(z)z/\bar{z}$ for a nonnegative measurable function ρ. The name of D_{μ,z_0} comes from the following important relation: If f is μ-conformal in Ω and if we write $z = z_0 + r\mathrm{e}^{\mathrm{i}\theta}$, then

$$\left|\frac{\partial f}{\partial \theta}(z)\right|^2 = r^2 D_{\mu,z_0}(z) J_f(z) \tag{A.2.7}$$

holds for almost all $z \in \Omega$, where J_f is the Jacobian of f. The quantity $D_{-\mu,z_0}$ is called the *radial dilatation* of μ at z_0 because it satisfies

$$\left|\frac{\partial f}{\partial r}(z)\right|^2 = D_{-\mu,z_0}(z) J_f(z).$$

We call D_{μ,z_0} and $D_{-\mu,z_0}$ directional dilatations.

Theorem A.9. *Let μ be a Beltrami coefficient in $\mathbb{C}$. Assume that for each $z_0 \in \widehat{\mathbb{C}}$, one of the following conditions holds for some positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$:*

1) $D_{\mu,z_0}(z) \le M$ *a.e. in* $B(z_0, r_0)$.
2) *There is a dominating factor $H = H_{z_0}$ of divergence type such that*

$$\int_{B(z_0,r_0)} \mathrm{e}^{H(\Delta)} \mathrm{d}\mathscr{A}(z) \le M$$

holds when $z_0 \in \mathbb{C}$, while the above condition is replaced by

$$\int_{B(\infty,r_0)} \mathrm{e}^{H(D_{\mu,0}(z))} \frac{\mathrm{d}\mathscr{A}(z)}{|z|^4} \le M$$

when $z_0 = \infty$.

Then the canonical approximating sequence f_n for μ forms a normal family with respect to the uniform convergence in $\widehat{\mathbb{C}}$ and every limit function f of this sequence is a self-homeomorphism of $\widehat{\mathbb{C}}$. Moreover, f admits the following modulus of continuity estimates according to cases (1) or (2) at each point z_0 with $|z_0| \le R_0$, where R_0 is an arbitrary positive number:

$$|f(z) - f(z_0)| \le C|z - z_0|^{1/M} \tag{A.2.8}$$

or

$$|f(z) - f(z_0)| \le C \exp\left\{ -\int_{1+c}^{2m+c} \frac{\mathrm{d}t}{2H^{-1}(t)} \right\}, \tag{A.2.9}$$

respectively, for $|z - z_0| < r_1$ and $0 < r_1 \le \min\{r_0, R_0\}$, where $m = \log(r_1/|z - z_0|)$, $c = \log(M/\pi r_1^2)$ and C is a constant depending only on μ and R_0.

Remark A.1. Note that Theorem A.9 does not say anything about the μ-conformal mapping; it discusses the behavior of the limit mapping only. However, the following conclusions can be reached. Let $\mathrm{Sing}(\mu)$ be the singular set of a Beltrami coefficient μ, i.e.,

$$\mathrm{Sing}(\mu) = \left\{ z_0 \in \widehat{\mathbb{C}} : \operatorname{ess}\limsup_{z \to z_0} |\mu(z)| = 1 \right\}.$$

Note that $\mathrm{Sing}(\mu)$ is a compact set in $\widehat{\mathbb{C}}$. Note also that $\mathrm{Sing}(\mu)$ can have positive area, although $|\mu|$ is always assumed to be less than 1 almost everywhere. By definition, the open set $\Omega = \widehat{\mathbb{C}} \setminus \mathrm{Sing}(\mu)$ can be exhausted by a sequence of open subsets Ω_n, $n = 1, 2, \ldots$, in such a way that $|\mu| \le 1 - 1/n$ a.e. in Ω_n. Since $\mu_k = \mu$ a.e. in Ω_n for all $k \ge n$, the limit f is μ-conformal in Ω_n for each n. Therefore, f is locally quasiconformal in Ω, in particular, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, the complex dilatation of f agrees with the given μ a.e. in Ω, and $f|_\Omega$ is unique up to the postcomposition by a conformal map. Further discussions on the existence of μ-conformal homeomorphisms and their regularity will be conducted in the next section.

The proof of the theorem is based on Proposition A.20, and an estimate on the moduli of ring domains under homeomorphisms in the Sobolev space $W^{1,1}_{\mathrm{loc}}$ is stated in the lemmas below.

The first result can essentially be found in [15, 198]. We give it under somewhat weaker assumptions.

Lemma A.18. *Let μ be a Beltrami coefficient on a domain Ω in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a μ-conformal embedding. Suppose that $D_{\mu,z_0}(z)$ is locally integrable in the annulus $A = A(z_0, r_1, r_2) \subset \Omega$. Then*

$$\operatorname{mod} f(A) \ge \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r\psi_\mu(r, z_0)}, \tag{A.2.10}$$

where

$$\psi_\mu(r,z_0) = \frac{1}{2\pi}\int_0^{2\pi} D_{\mu,z_0}(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta. \tag{A.2.11}$$

Proof. We may assume that $z_0 = 0$ and $\Omega = A = A(0,1,R)$. We may further assume that $A' = f(A) = A(0,1,R')$. Let ν be the positive Borel measure on A defined by $\nu(E) = \mathscr{A}(f(E))$. By a usual argument (see, e.g., [9, 152]), we obtain $J_f \mathrm{d}\mathscr{A} \le \mathrm{d}\nu$ on A in the sense of measure, where $J_f = |f_z|^2 - |f_{\bar{z}}|^2$. In particular, $J_f \in L^1_{\mathrm{loc}}(A)$.

Denote by γ_r the circle $|z| = r$. Then the assumption $f \in W^{1,1}_{\mathrm{loc}}(A)$ together with the Gehring–Lehto theorem (see [152]) implies that, for almost all $r \in (1,R)$, f is absolutely continuous on γ_r and totally differentiable at every point in γ_r except for a set of linear measure 0. By Fubini's theorem, we observe that D_μ and J_f are integrable on γ_r for almost all $r \in (1,R)$. For such an r, we have

$$2\pi \le \int_{\gamma_r} |\mathrm{d}\arg f| \le \int_{\gamma_r} \frac{|\mathrm{d}f(z)|}{|f(z)|} = \int_0^{2\pi} \frac{|f_\theta(r\mathrm{e}^{\mathrm{i}\theta})|}{|f(r\mathrm{e}^{\mathrm{i}\theta})|}\mathrm{d}\theta.$$

We use Schwarz's inequality and (A.2.7) to obtain

$$(2\pi)^2 \le r^2 \int_0^{2\pi} D_\mu(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta \int_0^{2\pi} \frac{J_f}{|f|^2}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta,$$

and hence,

$$\frac{2\pi}{r\psi_\mu(r)} \le r\int_0^{2\pi} \frac{J_f}{|f|^2}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta$$

for almost all $r \in (1,R)$, where $\psi_\mu(r) = \psi_\mu(r,0)$. Integrating both sides with respect to r from 1 to R, we obtain

$$\begin{aligned} 2\pi\int_1^R \frac{\mathrm{d}r}{r\psi_\mu(r)} &\le \int_1^R\int_0^{2\pi} \frac{J_f}{|f|^2} r\mathrm{d}\theta\mathrm{d}r = \int_A \frac{J_f \mathrm{d}\mathscr{A}}{|f|^2} \\ &\le \int_A \frac{\mathrm{d}\nu}{|f|^2} = \int_{A'} \frac{\mathrm{d}\mathscr{A}(w)}{|w|^2} = 2\pi\log R' = 2\pi \operatorname{mod} A' \end{aligned}$$

and thus arrive at the required inequality (A.2.10). □

The following auxiliary result may be of independent interest. The basic idea is due to Brakalova–Jenkins [56], Appendix A.1, see also [174, p. 51], Appendix A.4.

Lemma A.19. *Let f be a μ-conformal embedding of $A = A(z_0, r_0\mathrm{e}^{-m}, r_0)$ into $\mathbb{C}$. Suppose that a dominating factor H satisfies*

$$\int_A \mathrm{e}^{H(D_{\mu,z_0}(z))}\mathrm{d}\mathscr{A}(z) \le M, \quad \textit{if } z_0 \in \mathbb{C}, \tag{A.2.12}$$

and

$$\int_A \mathrm{e}^{H(D_{\mu,0}(z))} \frac{\mathrm{d}\mathscr{A}(z)}{|z|^4} \le M, \quad \text{if } z_0 = \infty. \tag{A.2.13}$$

Then we have

$$\operatorname{mod} f(A(z_0, r_0\mathrm{e}^{-m}, r_0)) \ge \int_{1/2}^{m} \frac{\mathrm{d}t}{H^{-1}(2t + \log(M/\pi r_0^2))}. \tag{A.2.14}$$

Proof. Let first $z_0 \neq \infty$. Setting

$$h(r) = \frac{r^2}{2\pi} \int_0^{2\pi} \mathrm{e}^{H(D_{\mu,z_0}(z_0+r\mathrm{e}^{\mathrm{i}\theta}))} \mathrm{d}\theta,$$

we rewrite inequality (A.2.12) in the form

$$2\pi \int_{r_0\mathrm{e}^{-m}}^{r_0} h(r)\frac{\mathrm{d}r}{r} = 2\pi \int_0^m h(r_0\mathrm{e}^{-t})\mathrm{d}t \le M.$$

By Chebyshev's inequality, the set $T = \{t \in (0,m) : h(r_0\mathrm{e}^{-t})) > L\}$ has length

$$|T| = \int_T \mathrm{d}t \le \frac{M}{2\pi L}.$$

Since e^H is a convex function, Jensen's inequality yields $\mathrm{e}^{H(\psi(r))} \le h(r)/r^2$ where $\psi(r) = \psi_\mu(r, z_0)$. This implies the inequality $\psi(r_0\mathrm{e}^{-t}) \le H^{-1}(2t + \log(L/r_0^2))$ for $t \in (0,m) \setminus T$. Lemma A.18 now yields

$$\begin{aligned}\operatorname{mod} f(A) &\ge \int_0^m \frac{\mathrm{d}t}{\psi(r_0\mathrm{e}^{-t})} \ge \int_{(0,m)\setminus T} \frac{\mathrm{d}t}{H^{-1}(2t + \log(L/r_0^2))} \\ &\ge \int_{|T|}^m \frac{\mathrm{d}t}{H^{-1}(2t + \log(L/r_0^2))} \ge \int_{M/2\pi L}^m \frac{\mathrm{d}t}{H^{-1}(2t + \log(L/r_0^2))}.\end{aligned}$$

Finally, letting $L = M/\pi$, we obtain (A.2.14).

The remaining case is that $z_0 = \infty$. Let $\varphi(z) = 1/z$ be the inversion. Let $\hat{\mu}$ be the complex dilatation of the map $g = \varphi \circ f \circ \varphi$. Then $D_{\hat{\mu},0}(z) = D_{\mu,0}(1/z)$. Now the required inequality immediately follows from the previous one. □

Proof of Theorem A.9. Let f_n be the canonical approximating sequence for μ. First, suppose that μ satisfies assumption (2) at $z_0 \in \mathbb{C}$. Since

$$H \circ D_{\mu_n,z_0}(z) \le \max\left\{H \circ D_{\mu,z_0}(z),\ H(1)\right\} \tag{A.2.15}$$

for the dominating factor H, the sequence

$$M_n = \int_{B(z_0,r_0)} \mathrm{e}^{H(D_{\mu_n,z_0}(z))} \mathrm{d}\mathscr{A}(z) \tag{A.2.16}$$

satisfies

$$\lim_{n\to\infty} M_n = \int_{B(z_0,r_0)} \mathrm{e}^{H(D_{\mu,z_0}(z))}\mathrm{d}\mathscr{A}(z) \le M = M(z_0)$$

by Lebesgue's dominating convergence theorem. Set $\widehat{M}_n = \sup\{M_k; k \ge n\}$. We now see from Lemma A.19 that

$$\operatorname{mod} f_n(A(z_0, r\mathrm{e}^{-m}, r)) \ge \int_{1/2}^{m} \frac{\mathrm{d}t}{H^{-1}(2t + \log(\widehat{M}_n/\pi r^2))} =: \rho_n(z_0, r\mathrm{e}^{-m}, r)$$

for $0 < r \le r_0$ and $m > 1/2$. Then, by virtue of Lemma A.17, $\rho_n(z_0, r\mathrm{e}^{-m}, r) \to \infty$ as $m \to +\infty$. For $z_0 = \infty$ the same conclusion follows. □

In the case when μ satisfies assumption (1), the situation is simpler. Since $\psi_\mu(r, z_0) \le M$ for a.e. $0 < r \le r_0$, by Lemma A.18, we obtain $\operatorname{mod} f_n(A(z_0, r\mathrm{e}^{-m}, r)) \ge m/M$ for $0 < r \le r_0$ and $m > 0$, so we just set $\rho_n(z_0, r\mathrm{e}^{-m}, r) = m/M$ in this case. For the other points (z_0, r, R), we simply set $\rho_n(z_0, r, R) = 0$.

The sequence f_k, $k \ge n$, then belongs to the class $\mathscr{H}_{\rho_n}$ for each n. By Proposition A.20, the limit map f of any convergent subsequence of f_k is homeomorphic. We show the estimates of the modulus of continuity of f. Since case (1) is easier to treat, we consider only case (2). Noting that $f \in \mathscr{H}_{\rho_n}$ for each n, we obtain $f \in \mathscr{H}_{\rho_\infty}$, where $\rho_\infty = \lim_{n\to\infty} \hat{\rho}_n$. By Proposition A.20, we now have

$$|f(z) - f(z_0)| \le C \exp\big(-\rho_\infty(z_0, r_1\mathrm{e}^{-m}, r_1)\big)$$

for $|z - z_0| \le r_1\mathrm{e}^{-m}$, $|z_0| \le R_0$ and $r_1 \le \min\{r_0(z_0), R_0\}$, where C is a constant depending only on μ and R_0. We now obtain (A.2.9) by

$$\rho_\infty(z_0, r_1\mathrm{e}^{-m}, r_1) \ge \int_{1/2}^{m} \frac{\mathrm{d}t}{H^{-1}(2t + \log(M/\pi r_1^2))} = \int_{1+c}^{2m+c} \frac{\mathrm{d}t}{2H^{-1}(t)},$$

where $c = \log(M/\pi r_1^2)$.

A.2.3 Existence Theorems

We first recall some regularity results (see, e.g., [56], Lemmas 4–6 and Proposition 10, and also [174, 185]) on the degenerate Beltrami equation and then use them and Theorem A.9 to study the existence problem of the solutions to the aforementioned equation.

Proposition A.21. *Let μ be a Beltrami coefficient in $\mathbb{C}$ and suppose that a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a uniform limit of the canonical approximating sequence f_n for μ. If $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 1$, then $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$, $q = 2p/(1+p)$, and f satisfies the Beltrami equation with μ. Moreover, $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.*

Remark A.2. It is shown in [56, Lemma 3] that $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$ implies the ACL property of f.

Remark A.3. The Sobolev embedding theorem for spheres [169] or Gehring's oscillation inequality (see, e.g., [116, Lemma 5.2]) implies that if $f:\mathbb{C}\to\mathbb{C}$ is a μ-conformal homeomorphism with $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$, then for every compact set E in $\mathbb{C}$, there exist constants C and a such that

$$|f(z_1)-f(z_2)| \geq C\,\mathrm{e}^{-a/|z_1-z_2|^2} \tag{A.2.1}$$

for $z_1, z_2 \in E$ with $z_1 \neq z_2$. The same inequality is also obtained in [247, p. 75] as a consequence of the length-area principle.

Proposition A.19 together with Theorem A.9 yields the following statement:

Theorem A.10. *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that:*

(a) $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ *for some* $p>1$.
(b) *For each point $z_0 \in \widehat{\mathbb{C}}$, there exist positive constants $M=M(z_0)$ and $r_0=r_0(z_0)$ so that either* (1) $D_{\mu,z_0} \leq M$ *a.e. in* $B(z_0,r_0)$ *or* (2)

$$\int_{B(z_0,r_0)} \mathrm{e}^{H(D_{\mu,z_0}(z))}\,\mathrm{d}\mathscr{A}(z) \leq M, \quad \textit{when } z_0\in\mathbb{C}, \tag{A.2.2}$$

and

$$\int_{B(\infty,r_0)} \mathrm{e}^{H(D_{\mu,0}(z))}\,\frac{\mathrm{d}\mathscr{A}(z)}{|z|^4} \leq M, \quad \textit{when } z_0=\infty, \tag{A.2.3}$$

holds for some dominating factor $H=H_{z_0}$ of divergence type.
Then there exists a normalized homeomorphic solution $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ of (A.2.1) such that $f\in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$, $q=2p/(1+p)$, and $f^{-1}\in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. This homeomorphism admits the modulus of continuity estimate, either $|f(z)-f(z_0)| \leq C|z-z_0|^{1/M}$ or

$$|f(z)-f(z_0)| \leq C\exp\left\{-\frac{1}{2}\int_{1+c}^{2m+c}\frac{\mathrm{d}t}{H_{z_0}^{-1}(t)}\right\}, \tag{A.2.4}$$

respectively, according to case (1) *or* (2)*, for $|z-z_0|<r_1$, $|z_0|\leq R_0$ and $r_1 \leq \min\{r_0(z_0),R_0\}$, where $m=\log(r_1/|z-z_0|)$, $c=\log(M(z_0)/\pi r_1^2)$, $R_0>0$ is a fixed number, and C is a constant depending only on μ and R_0.*

Proof. Let f_n be the canonical approximating sequence corresponding to the Beltrami coefficient μ. From Theorem A.9, we see that f_n forms a precompact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$ and every limit function f of this family is a self-homeomorphism of $\widehat{\mathbb{C}}$. Passing to a subsequence, we may assume that $f_n \to f$ uniformly in $\widehat{\mathbb{C}}$ and $\mu_n\to\mu$ a.e. as $n\to\infty$. Since $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$

for some $p > 1$, we see by Proposition A.21 that $f \in W^{1,q}_{\text{loc}}(\mathbb{C})$, $q = 2p/(1+p)$ and f satisfies (A.2.1). Moreover, $f_n^{-1} \to f^{-1}$ uniformly in $\widehat{\mathbb{C}}$ as $n \to \infty$ and $f^{-1} \in W^{1,2}_{\text{loc}}(\mathbb{C})$. The modulus of continuity estimate in (A.2.4) follows from Theorem A.9.
□

Remark A.4. Assumption (b) in Theorem A.10 implies $D_{\mu,z_0} \in L^p(B(z_0,r_0))$ for all $p > 1$. Assumption (a), however, cannot be dropped in order to have the regularity condition $f \in W^{1,1}_{\text{loc}}(\mathbb{C})$. Indeed, for $\mu(z) = (1+\mathrm{i}|z|^2)^{-1} z/\bar{z}$, $|z| < 1$ and $\mu(z) = 0$, $|z| > 1$, the normalized μ-conformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has the form $f(z) = z\mathrm{e}^{\mathrm{i}(1/|z|^2-1)}$, $|z| < 1$, and $f(z) = z$, $|z| \geq 1$. A simple calculation shows that $D_{\mu,0}(z) = 1$ a.e. in $\mathbb{C}$, and therefore, D_{μ,z_0} is bounded near z_0 for each $z_0 \in \widehat{\mathbb{C}}$. Hence, assumption (b) is fulfilled, but $|f_{\bar{z}}| = 1/|z|^2$ is not locally integrable near the origin, and thus $f \notin W^{1,1}_{\text{loc}}(\mathbb{C})$. Note that $K_\mu(z) \geq 1/|z|^4$ and hence $K_\mu \notin L^1_{\text{loc}}(\mathbb{C})$.

Note also that assumption (a) can be stated in terms of the radial dilatation of μ. In fact, we can replace it by the condition that $D_{-\mu,z_0}$ belongs to $L^p(B(z_0,r_0))$ for each finite z_0, where $p > 1$ is a fixed number. This follows from the double inequality

$$\frac{K_\mu}{2} \leq D_{\mu,z_0} + D_{-\mu,z_0} = \frac{1+|\mu|^2}{1-|\mu|^2} \leq K_\mu.$$

Specifying appropriate dominating factors $H(x)$, we obtain more concrete consequences from Theorem A.10. Typical choices for $H(x)$ are ηx and $\eta x/(1+\log^+ x)$ for a positive constant η. The integrability condition (A.2.2) now reduces to

$$\int_{B(z_0,r_0)} \mathrm{e}^{\eta D_{\mu,z_0}(z)} \mathrm{d}\mathscr{A}^\sharp(z) < +\infty v \tag{A.2.5}$$

and to the local subexponential integrability condition

$$\int_{B(z_0,r_0)} \exp\left\{ \frac{\eta D_{\mu,z_0}(z)}{1+\log^+ D_{\mu,z_0}(z)} \right\} \mathrm{d}\mathscr{A}^\sharp(z) < +\infty, \tag{A.2.6}$$

respectively. More examples will be given in Sect. A.2.3.

The following result can be viewed as an extended version of the corresponding existence theorems from [56, 116]:

Theorem A.11. *Let H be a dominating factor of divergence type. Suppose that a Beltrami coefficient μ on $\mathbb{C}$ with $\|\mu\|_\infty \leq 1$ satisfies*

$$\int_{\mathbb{C}} \mathrm{e}^{H(K_\mu(z))} \mathrm{d}\mathscr{A}^\sharp(z) < +\infty, \tag{A.2.7}$$

where $\mathrm{d}\mathscr{A}^\sharp(z) = (1+|z|^2)^{-2}\mathrm{d}\mathscr{A}(z)$ denotes the spherical area element. Then there exists a normalized homeomorphic solution f to the Beltrami equation such that $f \in W^{1,q}_{\text{loc}}(\mathbb{C})$ for every $1 \leq q < 2$ and that $f^{-1} \in W^{1,2}_{\text{loc}}(\mathbb{C})$.

Proof. Since H is of divergence type, assumption (A.2.7) implies that $K_\mu \in L^p_{\rm loc}(\mathbb{C})$ for every $1 < p < \infty$. On the other hand, the inequality $\Delta \le K(z)$ a.e. and the convergence of the integral (A.2.7) imply the local assumption (A.2.2). Hence, by Theorem A.10, we have the required conclusion. □

Remark A.5. Condition (A.2.7) is optimal for the solvability of Beltrami equations in the following sense:

Assume that H is a dominating factor of convergence type. We may assume that H is smooth enough and $H(1) = 1$. Then, by Theorem 3.1 of [116], there exists a μ satisfying (A.2.7) for which the following holds:

1. $\mu = 0$ off the unit disk B.
2. K_μ is locally essentially bounded in $\mathbb{C}\setminus\{0\}$.
3. There are no $W^{1,1}_{\rm loc}$-solutions to the Beltrami equation with μ in the unit disk which are continuous at the origin other than constant functions.
4. There is a solution f to the Beltrami equation in the weak-$W^{1,2}(B)$ Sobolev space, where weak-$W^{1,2}(B) = \bigcap_{1\le q<2} W^{1,q}(B)$, which maps the punctured disk $B\setminus\{0\}$ homeomorphically onto the annulus $A(0,1,R)$ for some $1 < R < +\infty$.

Corollary A.1. *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that, for some $\eta > 0$,*

$$\int_{\mathbb{C}} \mathrm{e}^{\eta K_\mu(z)} \mathrm{d}\mathscr{A}^\sharp(z) < +\infty. \tag{A.2.8}$$

Then there exists a μ-conformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f \in W^{1,q}_{\rm loc}(\mathbb{C})$ for every $q < 2$ and $f^{-1} \in W^{1,2}_{\rm loc}(\mathbb{C})$. Moreover, for every compact set $E \subset \mathbb{C}$, there are positive constants C, C' and a such that

$$C \exp\left(-\frac{a}{|z_1 - z_2|^2}\right) \le |f(z_1) - f(z_2)| \le C' \left|\log \frac{1}{|z_1 - z_2|}\right|^{-\eta/2} \tag{A.2.9}$$

for $z_1, z_2 \in E$ with $0 < |z_1 - z_2| < 1/e$. The exponent $-\eta/2$ is sharp.

The right-hand side of (A.2.9) can be written in the more precise form

$$|f(z_1) - f(z_2)| \le C \operatorname{dist}(E_0, E_1) \left(\frac{1 + \log(M/\pi R^2)}{\log 1/|z_1 - z_2|}\right)^{\eta/2}$$

for $z_1, z_2 \in B(0, R/2)$ with $|z_1 - z_2| < 1/e$, where

$$M = \int_{B(0,R)} \mathrm{e}^{\eta K_\mu} \mathrm{d}\mathscr{A},$$

$E_0 = f(\overline{B}(0,R/2)), E_1 = f(\widehat{\mathbb{C}} \setminus B(0,R))$ and C is a constant depending only on η.

The sharpness can be seen by the following examples: Let $f : \mathbb{C} \to \mathbb{C}$ be the radial stretching defined by

$$f(z) = \frac{z}{|z|}\left(1 + \frac{2}{\beta}\log\frac{1}{|z|}\right)^{-\beta/2} \tag{A.2.10}$$

for $|z| \le 1$ and $f(z) = z$ for $|z| > 1$. (This example was given in [116, Sect. 11.1].) Then $K(z) = 1 + (2/\beta)\log(1/|z|)$ for $|z| < 1$. Hence, integrability condition (A.2.8) holds for $\beta > \eta$ but not for $\beta \le \eta$. Since $|f(z) - f(0)| = (1 + 2\beta^{-1}\log 1/|z|)^{-\beta/2} \sim (\log 1/|z|)^{-\beta/2}$ as $z \to 0$, the exponent $\eta/2$ in (A.2.9) cannot be replaced by any larger number.

Remark A.6. It is noted in [25] that a necessary and sufficient condition for a measurable function $K(z) \ge 1$ to be majorized in $\Omega \subset \mathbb{C}$ by a function $M \in BMO(\mathbb{C})$ is that

$$\int_{\Omega} e^{\eta K(z)} d\mathscr{A}^{\sharp}(z) < +\infty \tag{A.2.11}$$

for some positive number η. Moreover, M can be chosen so that $\|M\|_{\mathrm{BMO}} \le C/\eta$ where C is an absolute constant. For mappings of BMO-bounded distortion, David [70] has proved the estimate

$$|f(z_1) - f(z_2)| \le A\left|\log\frac{1}{|z_1 - z_2|}\right|^{-b/\|M\|_{\mathrm{BMO}}} \tag{A.2.12}$$

for some positive constant b, which agrees with our estimate (A.2.9).

Remark A.7. It is shown in [116] that if μ has a compact support, then there exists a number $\eta_0 > 1$ such that the Beltrami equation for μ satisfying (A.2.8) with $\eta \ge \eta_0$ admits a unique principal solution f with $f(z) - z \in W^{1,2}(\mathbb{C})$.

The following consequence is due to [116, Theorem 14.2] except for the modulus of continuity estimate. The almost same result was earlier obtained in [56] (see Remark A.8 below).

Corollary A.2. *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that*

$$\int_{\mathbb{C}} \exp\left\{\frac{\eta K_\mu(z)}{1 + \log K_\mu(z)}\right\} d\mathscr{A}^{\sharp}(z) < \infty \tag{A.2.13}$$

for some $\eta > 0$. Then there exists a homeomorphic solution f of (A.2.1) in $\widehat{\mathbb{C}}$ such that $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ for every $q < 2$ and $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Moreover, for every compact set $E \subset \mathbb{C}$, there are constants C, C' and a such that

$$C\exp\left\{-\frac{a}{|z_1 - z_2|^2}\right\} \le |f(z_1) - f(z_2)| \le C'\left(\log\log\frac{1}{|z_1 - z_2|}\right)^{-\eta/2} \tag{A.2.14}$$

for $z_1, z_2 \in E$ with $0 < |z_1 - z_2| < e^{-e}$. The exponent $-\eta/2$ is sharp.

The modulus of continuity follows from the fact that $\eta^{-1}y\log y < H^{-1}(y)$ for sufficiently large y where $H(x) = \eta x/(1+\log^+ x)$. More precisely, we have an estimate in the form

$$|f(z_1)-f(z_2)| \le C\,\mathrm{dist}(E_0,E_1)\left(\frac{\log\log(M/\pi R^2)}{\log\log(1/|z_1-z_2|)}\right)^{\eta/2}$$

for $z_1,z_2 \in B(0,R/2)$ with $|z_1-z_2| < \mathrm{e}^{-e}$, where

$$M = \int_{B(0,R)} \exp\left\{\frac{\eta K_\mu(z)}{1+\log K_\mu(z)}\right\} \mathrm{d}\mathscr{A}(z),$$

$E_0 = f(\overline{B}(0,R/2)), E_1 = f(\widehat{\mathbb{C}} \setminus B(0,R))$ and C is a constant depending only on η.

Next, we show the sharpness by examples. Let $f : \mathbb{C} \to \mathbb{C}$ be the radial stretching defined by

$$f(z) = \frac{z}{|z|}\left(1+\frac{2}{\beta}\log\log\frac{e}{|z|}\right)^{-\beta/2} \qquad \text{(A.2.15)}$$

for $|z| \le 1$ and $f(z) = z$ for $|z| > 1$. Then

$$K(z) = \log\frac{e}{|z|}\left(1+\frac{2}{\beta}\log\log\frac{e}{|z|}\right) \qquad \text{for } |z| < 1.$$

(This example appeared in [116, Sect. 12.1]; however, there seems to be an error in the formula for $K(z)$.) Hence,

$$\frac{K(z)}{1+\log K(z)} = \frac{2}{\beta}\log\frac{e}{|z|}\left(1-\frac{\log_3 e/|z|}{\log_2 e/|z|}+O\left(\frac{1}{\log_2 e/|z|}\right)\right)$$

as $z \to 0$ and, in particular, we see that $\exp\{\eta K(z)/(1+\log K(z))\}$ is integrable in $|z| < 1$ if and only if $\beta \ge \eta$. Therefore, the exponent $\eta/2$ in (A.2.14) cannot be replaced by any larger number in general.

Remark A.8. The first proof of the above statement was given in [56] under the assumption of special behavior of $K_f(z)$ around the point at infinity:

$$\int_{B(0,R)} K(z)\mathrm{d}\mathscr{A}(z) = O(R^2).$$

This assumption says that K is bounded in the sense of the mean, and hence, it is different from (A.2.13). Theorem 14.2 in [116] contains also the quite accurate regularity assertion

$$\int_{\mathbb{C}} \frac{|\Psi(z)|^2}{\log(e+\Psi(z))\log_2(3+\Psi(z))}\mathrm{d}\mathscr{A}^\sharp(z) < +\infty,$$

where $\Psi(z)$ stands for the spherical derivative $\frac{1+|z|^2}{1+|f(z)|^2}|Df(z)|$ of f.

Next, we prove Theorem A.7 given in the Introduction, which shows that if $\mathrm{Sing}(\mu)$ of μ is a finite set, then μ can be modified without changing its absolute value and the modified μ admits a "good" solution.

Proof. Let $\mathrm{Sing}(\mu) = \{a_1, \ldots, a_n\}$. Without loss of generality, we will assume that all a_k's are finite. Let δ be a positive number such that the disks $|z - a_k| \le \delta$, $k = 1, \ldots, n$, are disjoint. Setting

$$\mu'(z) = |\mu(z)| \cdot \frac{z - a_k}{\bar{z} - \bar{a}_k}, \qquad k = 1, \ldots, n,$$

for $|z - a_k| \le \delta$, and $\mu'(z) = \mu(z)$ otherwise, we see that $D_{\mu',z_0}(z)$ is essentially bounded in a sufficiently small disk $B(z_0, r_0)$ for each $z_0 \in \widehat{\mathbb{C}}$. By Theorem A.9 and Remark A.1, there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}$, and $f|_\Omega$ satisfies the Beltrami equation with μ'. The condition $|\mu'(z)| = |\mu(z)|$, which holds a.e., is immediate by construction. □

Remark A.9. For every $H \in \mathscr{D}$, there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ locally quasiconformal in $\widehat{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}$ with complex dilatation μ such that

$$\int_{\mathbb{C}} \mathrm{e}^{H(K_\mu(z))} \mathrm{d}\mathscr{A}^\sharp(z) = +\infty,$$

but weaker directional condition (A.2.2) holds.

In the above examples, the singular set of μ consists of isolated points only. In the following examples, the singular sets of μ are the whole extended real axis $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

Example A.1. Define the Beltrami coefficient μ in $\widehat{\mathbb{C}}$ for $\alpha \in [1,2)$ as

$$\mu(z) = 1 - \frac{2}{\alpha \log \frac{1}{|y|}}, \qquad z = x + \mathrm{i}y,$$

for $|y| \le y_0$, and as $\mu(z) = 0$ for $|y| > y_0$, where $0 < y_0 \le \mathrm{e}^{-1/\alpha}$ is a constant. Then integrability condition (A.2.8) with $\eta = 1$ does not hold in any neighborhood of each point on the extended real axis, whereas condition (A.2.5) with $\eta = 1$ still holds. Indeed, since $K_\mu(z) = 1 + \alpha \log(1/|y|)$ for $|y| < y_0$, we see that $\mathrm{e}^{K_\mu(z)} = \mathrm{e}|y|^{-\alpha}$. Now the assumption $\alpha \ge 1$ implies that e^{K_μ} is not locally integrable at any point on the real axis. On the other hand, writing $z = r\mathrm{e}^{\mathrm{i}\theta}$, we compute for any $a \in \mathbb{R}$

$$D_{\mu,a}(z+a) = D_{\mu,0}(z) = \frac{1-\mu}{1+\mu} + \frac{4\mu}{1+\mu} \cdot \frac{\sin^2\theta}{1-\mu} \le 1 + \alpha \sin^2\theta \left(\log \frac{1}{|y|}\right),$$

where the inequality $4\mu/(1+\mu) \le 2$ has been used. Since the function $x \log x$ is bounded in $(0,1)$, we obtain

$$\exp(D_{\mu,a}(z+a)) \le \mathrm{e}|y|^{-\alpha\sin^2\theta} \le \mathrm{e}r^{-\alpha}\mathrm{e}^{-\frac{\alpha}{2}\sin^2\theta\log(\sin^2\theta)} \le Cr^{-\alpha}$$

for some constant C. Hence, for sufficiently small $r_0 > 0$, we have

$$\int_{B(a,r_0)} \mathrm{e}^{D_{\mu,a}(z)}\mathrm{d}\mathscr{A}^\sharp(z) \le 2\pi C\int_0^{r_0} r^{1-\alpha}\mathrm{d}r < +\infty.$$

Thus, condition (A.2.5) holds for each $a \in \mathrm{Re}$. Similarly, we verify (A.2.5) for $a = \infty$.

Example A.2. Another example of Beltrami coefficient μ in $\mathbb{C}$, for which $\mathrm{Sing}(\mu) = \widehat{\mathbb{R}}$, is provided by

$$\mu(z) = 1 - \frac{2}{\alpha\log\frac{1}{|y|}\log\log\frac{1}{|y|}}, \quad z = x + \mathrm{i}y,$$

for $|y| \le y_0$, and $\mu(z) = 0$ for $|y| > y_0$, where α is a constant with $1 < \alpha < 2e/(e+1) = 1.462\ldots$ and y_0 is a positive constant with $y_0 \le \mathrm{e}^{-e^{1/\alpha}}$. An elementary computation shows that integrability assumption (A.2.13) does not hold at any neighborhood of each point lying on the real axis. On the other hand, assumption (A.2.6) still holds. Indeed, since $K_\mu(z) = 1 + \alpha\log(1/|y|)\log_2(1/|y|)$, $|y| < y_0$, we obtain

$$\frac{K_\mu(z)}{1+\log K_\mu(z)} = \frac{\alpha\log\frac{1}{|y|}\log_2\frac{1}{|y|}}{\log_2\frac{1}{|y|} + \log_3\frac{1}{|y|} + O(1)} = \alpha\log\frac{1}{|y|}\left[1 + O\left(\frac{\log_3\frac{1}{|y|}}{\log_2\frac{1}{|y|}}\right)\right]$$

as $y \to 0$. In particular, we see that $\mathrm{e}^{K_\mu(z)/(1+\log K_\mu(z))}$ is not locally integrable at every point on the real axis. On the other hand, as in the previous case, we obtain

$$D_{\mu,0}(z) \le 1 + \alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}.$$

Hence, under the assumption that $\alpha\sin^2\theta\log(1/|y|)\log_2(1/|y|)$ is large enough, we have

$$\begin{aligned}
&\frac{D_{\mu,0}(z)}{1+\log^+ D_{\mu,0}(z)} \\
&\quad\le \frac{1+\alpha\sin^2\theta\log\frac{1}{|y|}\left[\log\left(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}\right) - \log\left(\alpha\sin^2\theta\log_2\frac{1}{|y|}\right)\right]}{1+\log\left(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}\right)} \\
&\quad= 1+\alpha\sin^2\theta\log\frac{1}{|y|} - \frac{\alpha\sin^2\theta\log\frac{1}{|y|}\left[\log\left(\alpha\sin^2\theta\right)+\log_3\frac{1}{|y|}\right]}{1+\log\left(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}\right)}
\end{aligned}$$

$$\leq \alpha \sin^2\theta \log\frac{1}{|y|} + \frac{\alpha e^{-1}\log\frac{1}{|y|}}{1+\log\left(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}\right)}$$

$$\leq \alpha\left(\sin^2\theta + e^{-1}\right)\log\frac{1}{|y|},$$

where we used the inequality $-x\log x \leq e^{-1}$ for $0 < x$. This yields

$$\exp\left(\frac{D_{\mu,0}(z)}{1+\log^+ D_{\mu,0}(z)}\right) \leq |y|^{-\alpha(\sin^2\theta+e^{-1})} \leq Cr^{-\alpha(1+e^{-1})}|\sin\theta|^{-\alpha/e}$$

for some positive constant C. The last function in polar coordinates (r,θ) is integrable with respect to the measure $r\mathrm{d}\theta\mathrm{d}r$ over $0 < r < 1, 0 < |\theta| < \pi$ because $\alpha(1+e^{-1}) - 1 < 1$ and $\alpha/e < 1$. Thus, the function $\exp(D_{\mu,a}/(1+\log^+ D_{\mu,a}))$ is integrable in a neighborhood of each $a \in \mathrm{Re}$. Similarly, we obtain the same assertion for $a = \infty$. Therefore, condition (A.2.6) holds for each point $z_0 \in \widehat{\mathbb{C}}$.

Remark A.10. For the above examples, whose Beltrami coefficients $\mu(z) = \mu(x+\mathrm{i}y)$ depend on y only, we can give normalized μ-conformal homeomorphisms $f:\mathbb{C}\to\mathbb{C}$ in the explicit form

$$f(z) = x + \mathrm{i}\int_0^y \frac{1-\mu(it)}{1+\mu(it)}\mathrm{d}t.$$

A.2.4 Uniqueness

The following remark is a simple consequence of a well-known removability theorem for analytic functions:

Proposition A.22. *Let μ be a Beltrami coefficient in a domain Ω such that the singular set $E = \mathrm{Sing}(\mu)$ in Ω is countable. Suppose that a topological embedding $f:\Omega\to\widehat{\mathbb{C}}$ is locally quasiconformal in $\Omega\setminus E$ and satisfies (A.2.1). Then f has the property that for any homeomorphic solution $\hat f$ of (A.2.1) in $\Omega\setminus E$, there exists a conformal map h in $\Omega' = f(\Omega)$ such that $\hat f = h\circ f$ in $\Omega\setminus E$. In particular, $\hat f$ extends to an embedding of Ω.*

Indeed, let us assume that $\hat f$ is another homeomorphic solution to the Beltrami equation (A.2.1) in $\Omega\setminus E$. Note that $\|\mu\|_{L^\infty(\Omega_0)} < 1$ for every relatively compact open subset Ω_0 of $\Omega\setminus E$. The local integrability of the Jacobian $J_{\hat f} = (1-|\mu|^2)|\hat f_z|^2$ of $\hat f$ (cf. the proof of Lemma A.18) implies that $\hat f \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus E)$. Since $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega'\setminus f(E))$, the function $h = \hat f\circ f^{-1}$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega'\setminus f(E))$ and satisfies $h_{\bar z} = 0$ there. Now, Weyl's lemma yields that h is an injective holomorphic map in $\Omega'\setminus f(E)$. Since $f(E)$ is closed and countable in Ω', it is removable for such a function, and we conclude that h can be extended to a conformal map on Ω'.

In order to apply the removability arguments for the uniqueness problem, it seems necessary to have information on the singular set E and its image $E' = f(E)$ simultaneously; see, e.g., [25,133]. As Lehto [149] noted, it is reasonable to consider totally disconnected sets E. However, this condition does not imply uniqueness in general. We will introduce a geometric condition on totally disconnected compact sets E and show that this condition can be combined with integrability conditions to guarantee the uniqueness.

A positive function $m(r)$ defined on the interval $(0,\delta)$ for small $\delta > 0$ is said to be a *modulus bound* for a dominating factor H of divergence type if it satisfies the condition

$$\liminf_{r\to 0} \int_0^{m(r)} \frac{\mathrm{d}t}{H^{-1}(2t - 2\log r)} > 0. \tag{A.2.1}$$

Note that $m(r) \to +\infty$ as $r \to 0$.

For example, if $H(x) = \eta x$, then $m(r) = \varepsilon \log(1/r)$ is a modulus bound for H, where ε is an arbitrary positive constant. If $H(x) = \eta x/(1+\log^+ x)$, then $m(r) = \varepsilon(\log(1/r))^C$ is a modulus bound for H, where $\varepsilon > 0$ and $C > 1$ are arbitrary constants. More examples will be given in Sect. A.2.3. Each modulus bound $m(r)$ generates the family of annuli $A_m(z_0,r) = A(z_0, r\mathrm{e}^{-m(r)}, r)$ around every point $z_0 \in \widehat{\mathbb{C}}$.

The following notions describe the thinness of the boundary: Let H be a dominating factor. A compact subset E of $\widehat{\mathbb{C}}$ is said to be *H-coarse* at $z_0 \in E$ if there exists a modulus bound $m(r)$ for H such that, for any small number $\delta > 0$, there is an r with $0 < r < \delta$ such that $A_m(z_0,r) \cap E = \emptyset$. The set E is said to be H-coarse if E is H-coarse at each point $z_0 \in E$. We will also say that E is *radially coarse* at $z_0 \in E$ if a positive constant function can be chosen as $m(r)$ above; more precisely, there exists a positive number m such that, for any small number $\delta > 0$, one can choose an $0 < r < \delta$ with $A(z_0, r\mathrm{e}^{-m}, r) \cap E = \emptyset$.

It is clear that $z_0 \in E$ forms a degenerate boundary component of $\Omega = \widehat{\mathbb{C}} \setminus E$ if E is H-coarse at the point.

Theorem A.12. *Let μ be a Beltrami coefficient in $\widehat{\mathbb{C}}$ such that $E = \mathrm{Sing}(\mu)$ is a totally disconnected compact subset of $\widehat{\mathbb{C}}$. Assume that one of the following conditions holds for each $z_0 \in E$:*

1) *$D_{\mu,z_0}(z)$ is essentially bounded in a neighborhood of z_0 and E is radially coarse at z_0.*
2) *There is a dominating factor $H = H_{z_0}$ of divergence type for which E is H-coarse at z_0 and*

$$\int_V \mathrm{e}^{H(D_{\mu,z_0}(z))} \mathrm{d}\mathscr{A}^\sharp(z) < +\infty \tag{A.2.2}$$

for some open neighborhood V of z_0.

Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus E$ and satisfies the Beltrami equation with μ in Ω. If $\hat{f}$ is another *homeomorphic solution to the Beltrami equation in Ω with the same μ, then $\hat{f} = h \circ f$ for a Möbius transformation h. In particular, $\hat{f}$ can be extended to a homeomorphism of $\widehat{\mathbb{C}}$.*

For the proof of this theorem, we need the following auxiliary lemma which is due to Gotoh and Taniguchi [96]. A compact set E is said to be *annularly coarse* at $z_0 \in E$ if we can find a nesting sequence of disjoint ring domains A_n around z_0 in $\widehat{\mathbb{C}} \setminus E$ with $\inf_n \operatorname{mod} A_n > 0$. Here, a sequence of disjoint ring domains A_n is said to be nesting if each A_n separates A_{n-1} from A_{n+1}. A compact set E is said to be annularly coarse if E is annularly coarse at every point in E.

Lemma A.20. *(Gotoh–Taniguchi). If a compact subset E of the Riemann sphere is annularly coarse, then E is removable for conformal mappings off E.*

Proof of Theorem A.12. Let f be a uniform limit of the canonical approximating sequence for μ. Note that the existence of such an f is guaranteed by Theorem A.9. Set $E' = f(E)$. We show that E' is annularly coarse. We may assume that $E \subset \mathbb{C}$. Let $z_0 \in E$. Suppose first that case (2) occurs for z_0. Then, by assumption, there is an $H \in \mathscr{D}$ and a constant M with

$$\int_{B(z_0,r_0)} \mathrm{e}^{H(D_{\mu,z_0}(z))} \mathrm{d}\mathscr{A}(z) \le M$$

for which E is H-coarse at z_0 with modulus bound $m : (0,\delta) \to (0,+\infty)$. We then take arbitrarily small $r > 0$ so that $A_m(z_0,r) \subset \Omega$. By Lemma A.19, we estimate

$$\operatorname{mod} f(A_m(z_0,r)) \ge \int_{1/2}^{m(r)} \frac{\mathrm{d}t}{H^{-1}(2t + \log(M/\pi r^2))}.$$

It is easy to see that

$$\int_a^b \frac{\mathrm{d}t}{H^{-1}(2t + c - 2\log r)} \to 0$$

as $r \to 0$ for any fixed a, b, and c. Hence, by the definition of the modulus bound, we obtain

$$\liminf_{r\to 0} \int_{1/2}^{m(r)} \frac{\mathrm{d}t}{H^{-1}(2t + \log(M/\pi r^2))} = \liminf_{r\to 0} \int_0^{m(r)} \frac{\mathrm{d}t}{H^{-1}(2t - 2\log r)} > 0.$$

We can now find a sequence r_n with $0 < r_{n+1} < r_n \mathrm{e}^{-m(r_n)}$ such that $A_m(z_0,r_n) \subset \Omega$ for $n = 1,2,\ldots$ and that $\inf_n \operatorname{mod} f(A_m(z_0,r_n)) > 0$. Hence, we conclude that E' is annularly coarse at $f(z_0)$.

If case (1) occurs for z_0, then E is radially coarse at z_0. Now, it easy to see that E' is annularly coarse at $f(z_0)$ (cf. Lemma A.18).

Assume that $\hat{f}$ is another homeomorphic solution to the Beltrami equation (A.2.1) with μ in Ω. Then the function $h = \hat{f} \circ f^{-1}$ is an injective holomorphic function in $\widehat{\mathbb{C}}\setminus E'$. Since $E' = f(E)$ is removable for such a function by Lemma A.20, we conclude that h extends to a Möbius transformation. Thus, the proof is complete. □

Specifying $H \in \mathscr{D}$, we obtain some consequences of Theorem A.12. We only consider the dominating factors $H(x) = \eta x$ and $H(x) = \eta x/(1+\log^+ x)$ corresponding to exponential and subexponential integrability assumptions on $K_\mu(z)$, respectively.

Corollary A.3. *Let E be a totally disconnected x-coarse compact subset of $\widehat{\mathbb{C}}$. Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ with $\mathrm{Sing}(\mu) \subset E$ such that*

$$\int_{\mathbb{C}} e^{\eta K_\mu(z)} d\mathscr{A}^\sharp(z) < \infty \tag{A.2.3}$$

holds for a positive constant η. Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus E$ and satisfies the Beltrami equation with μ in Ω. If $\hat{f} : \Omega \to \widehat{\mathbb{C}}$ is a locally quasiconformal embedding whose Beltrami coefficient agrees with μ a.e., then $\hat{f} = h \circ f$ for a Möbius transformation h. In particular, $\hat{f}$ extends to a homeomorphism of the Riemann sphere.

Corollary A.4. *Let E be a totally disconnected compact subset of $\widehat{\mathbb{C}}$ which is $x/(1+\log^+ x)$ coarse in E. Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ with $\mathrm{Sing}(\mu) \subset E$ such that*

$$\int_{\mathbb{C}} \exp\left\{\frac{\eta K_\mu(z)}{1+\log K_\mu(z)}\right\} d\mathscr{A}^\sharp(z) < \infty. \tag{A.2.4}$$

Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus E$ and satisfies the Beltrami equation with μ in Ω. If $\hat{f} : \Omega \to \widehat{\mathbb{C}}$ is a locally quasiconformal embedding whose Beltrami coefficient agrees with μ a.e., then $\hat{f} = h \circ f$ for a Möbius transformation h. In particular, $\hat{f}$ extends to a homeomorphism of the Riemann sphere.

A.2.5 Dominating Factors and Modulus Bounds

In this section, we provide critical dominating factors H of divergence type and their modulus bounds. The functions presented below are more or less standard. For example, similar examples can be found in [116].

We define the functions $\log_n, \exp_n, \Pi_{n,\alpha}$ and Π_n for $n = 0, 1, 2, \ldots$ and for $\alpha > 0$ by

$$\log_0 x = x, \quad \log_n x = \log(\log_{n-1} x) \quad (n = 1, 2, \ldots),$$
$$\exp_0 x = x, \quad \exp_n = \exp(\exp_{n-1} x) \quad (n = 1, 2, \ldots),$$

$$\Pi_{n,\alpha}(x) = x(\log_1 x)\cdots(\log_{n-1} x)(\log_n x)^{\alpha}, \text{ and}$$
$$\Pi_n(x) = \Pi_{n,1}(x) \quad (n = 0,1,2,\ldots).$$

In particular, $\log_1$ and $\exp_1$ coincide with the standard log and exp, respectively. Note that $\log_n$ is the inverse function of $\exp_n$ for each n. We also define the numbers e_n by $e_n = \exp_n 0$. Then for each $n > 0$, the functions $\log_n, \Pi_{n,\alpha}$, and Π_n are defined on $(e_{n-1}, +\infty)$, positive on $(e_n, +\infty)$, and greater than 1 on $(e_{n+1}, +\infty)$.

We consider the function $H(x) = x^2/\Pi_{n,\alpha}(x)$ for fixed $n \geq 0$ and $\alpha > 0$. Then $H'(x) = (1+o(1))H(x)/x$ and $H''(x) = (1+o(1))H(x)/\log x$ when $x \to +\infty$. In particular, $(\mathrm{e}^H)''/\mathrm{e}^H = H'' + (H')^2 = (1+o(1))(H')^2$. Hence, we can choose a sufficiently large number $x_{n,\alpha} > e_n$ so that the function $H_{n,\alpha}$ defined by

$$H_{n,\alpha}(x) = \begin{cases} H(x) = \dfrac{x^2}{\Pi_{n,\alpha}(x)} & \text{if } x > x_{n,\alpha} \\ H(x_{n,\alpha}) & \text{if } 0 \leq x \leq x_{n,\alpha} \end{cases}$$

is a dominating factor. Furthermore, $H_{n,\alpha}$ is of divergence type if and only if $\alpha \leq 1$, since the integral

$$\int_{x_{n,\alpha}}^{\infty} \frac{H_{n,\alpha}(x)\mathrm{d}x}{x^2} = \int_{x_{n,\alpha}}^{\infty} \frac{\mathrm{d}x}{\Pi_{n,\alpha}(x)} = \int_{x_{n,\alpha}}^{\infty} \frac{\mathrm{d}x}{x(\log x)\ldots(\log_{n-1} x)(\log_n x)^{\alpha}}$$

is divergent if and only if $\alpha \leq 1$. Note that the choice of $x_{n,\alpha}$ does not affect the integrability condition such as $\int_V \mathrm{e}^{H(D_{\mu,z_0})}\mathrm{d}\mathscr{A}^\sharp < +\infty$. We also write $H_n = H_{n,1}$.

Remark A.11. The dominating factor $H_0 = H_{0,1}$ is essentially the function x. The dominating factor $H_1 = H_{1,1}$ is equivalent to $H(x) = x/(1+\log^+ x)$ in the sense that for any $\eta > 1$, we have $H(x) < H_1(x) < \eta H(x)$ for sufficiently large x.

First, we give information on the behavior of the inverse function of $H_{n,\alpha}$.

Lemma A.21. *For each $n \geq 1, \alpha \in \mathbb{R}$ and $c > 0$, the estimate*

$$(cH_{n,\alpha})^{-1}(y) = \frac{\Pi_{n,\alpha}(y)}{c}\left(1 + (b+o(1))\frac{\log_2 y}{\log y}\right) \quad (y \to +\infty)$$

holds, where $b = 1$ when $n > 1$ and $b = \alpha^2$ when $n = 1$. Suppose further that $\alpha \leq 1$ and, in addition, if $n = 1$, suppose that $0 \leq \alpha$. Then

$$\int_{t_1}^{t} \frac{\mathrm{d}y}{(cH_{n,\alpha})^{-1}(y)} = \begin{cases} \dfrac{c}{1-\alpha}(\log_n t)^{1-\alpha} + O(1) & \textit{when } \alpha < 1 \\ c\log_{n+1} t + O(1) & \textit{when } \alpha = 1 \end{cases}$$

as $t \to +\infty$ *for a sufficiently large number* t_1. *When* $n = 1$ *and* $\alpha < 0$,

$$\int_{t_1}^{t} \frac{\mathrm{d}y}{(cH_{1,\alpha})^{-1}(y)} = \left(\frac{c}{1-\alpha} + o(1)\right)(\log t)^{1-\alpha}$$

as $t \to +\infty$ *be a sufficiently large number* t_1.

Proof. We assume that $n \geq 2$. (The same and somewhat easier method can be used for $n = 1$.) Letting $\eta = \eta(y)$ be a positive quantity with $\log \eta(y) = O(1)$ as $y \to \infty$, we observe

$$\begin{aligned}\log\left(\eta \Pi_{n,\alpha}(y)\right) &= \log \eta + \log y + \log_2 y + \cdots + \log_n y + \alpha \log_{n+1} y \\ &= \left(1 + (1+o(1))\frac{\log_2 y}{\log y}\right)\log y \quad (y \to +\infty).\end{aligned}$$

Inductively, we then obtain that

$$\log_k(\eta \Pi_{n,\alpha}(y)) = \log_k y \cdot (1 + (1+o(1))\log_2 y / \log y \log_2 y \cdots \log_k y)$$

for every integer $k \geq 1$. Therefore,

$$\begin{aligned}cH_{n,\alpha}\left(\eta \Pi_{n,\alpha}(y)\right) &= \frac{c\eta \Pi_{n,\alpha}(y)}{\log(\eta \Pi_{n,\alpha}(y)) \cdots (\log_n(\eta \Pi_{n,\alpha}(y)))^{\alpha}} \\ &= c\eta y\left(1 - (1+o(1))\frac{\log_2 y}{\log y}\right) \qquad \text{(A.2.5)}\end{aligned}$$

as $y \to +\infty$. We now restrict ourselves to the case when $c\eta = 1 + p\log_2 y/\log y$ for some constant p. Then $(1 - \log_2 y/\log y)c\eta = 1 + (p-1)(1+o(1))\log_2/\log y$, and since $\eta \Pi_{n,\alpha}(y) > {H_{n,\alpha}}^{-1}(y)$ for sufficiently large y if $p > 1$ and $\eta \Pi_{n,\alpha}(y) < {H_{n,\alpha}}^{-1}(y)$ for sufficiently large y if $p < 1$, the asymptotic formula for the inverse function follows.

We next show the second assertion. For simplicity, we assume that $\alpha < 1$; the case $\alpha = 1$ can be absorbed in this case because $H_{n,1}(y) = H_{n+1,0}(y)$ for large y. By the first assertion, we obtain

$$\int_{t_1}^{t} \frac{\mathrm{d}y}{(cH_{n,\alpha})^{-1}(y)} = \int_{t_1}^{t} \frac{c\mathrm{d}y}{\Pi_{n,\alpha}(y)} + (-bc + o(1))\int_{t_1}^{t} \frac{\log_2 y \mathrm{d}y}{y(\log y)^2 \log_2 y \cdots (\log_n y)^{\alpha}}$$

if $n > 1$. For $n = 1$, a slight modification is needed. Since the second integral in the right-hand side is convergent as $t \to +\infty$ under the assumptions on α, the required asymptotic formula is obtained. □

Proof of Theorem A.8. Without loss of generality, we may assume that $\alpha < 1$ (see the proof of Lemma A.21). Under the hypothesis, the integrability condition

$\int_{\mathbb{C}} e^{cH_{n,\alpha}(K_\mu(z))} d\mathscr{A}^\sharp(z) < \infty$ holds. Since $cH_{n,\alpha}$ is a dominated factor of divergence type, we conclude from Theorem A.11 the existence and the regularity of the normalized homeomorphic solution f to the Beltrami equation. We now investigate the modulus of continuity of f. From the assumption, for an arbitrary number $R > 1$, it follows that

$$\int_{B(z_0,1)} e^{cH_{n,\alpha}(K_\mu(z))} d\mathscr{A}(z) \le M = \int_{B(0,R+1)} e^{cH_{n,\alpha}(K_\mu(z))} d\mathscr{A}(z) < \infty$$

for each point z_0 with $|z_0| \le R$. Setting $b = \log(M/\pi)$, we obtain from Theorem A.10 that

$$|f(z) - f(z_0)| \le C \exp\left\{ -\frac{1}{2} \int_{1+b}^{2m+b} \frac{dt}{(cH_{n,\alpha})^{-1}(t)} \right\}, \quad |z - z_0| < 1,$$

where $m = \log(1/|z - z_0|)$ and C is a constant depending only on μ, c, n, α, and R. Applying Lemma A.21, in the case when $n > 1$, we have

$$\begin{aligned} \frac{1}{2} \int_{1+b}^{2m+b} \frac{dt}{(cH_{n,\alpha})^{-1}(t)} &= \frac{c}{2(1-\alpha)} (\log_n(2m+b))^{1-\alpha} + O(1) \\ &= \frac{c}{2(1-\alpha)} (\log_n(m))^{1-\alpha} + O(1) \end{aligned}$$

as $m \to +\infty$. The case when $n = 1$ follows similarly from the latter part of Lemma A.21. Thus, we have shown the estimate for the modulus of continuity of f.

Next, we construct an example; see the discussion after Theorem A.8. As we noted, a more abstract approach for a dominated factor of convergence type can be found in [116, Theorem 3.1].

Let $M(t) = \eta \log(1/t) \cdots \log_n(1/t)(\log_{n+1}(1/t))^\alpha = \eta \Pi_{n,\alpha}(\log(1/t))$, where $\eta > 0, n \ge 0$ and $\alpha \in \mathbb{R}$. Let f be the radial stretching

$$f(z) = \frac{z}{|z|} \exp \int_{\delta_0}^{|z|} \frac{dt}{tM(t)}$$

in $0 < |z| < \delta_0$, where δ_0 is a sufficiently small number. Then, as is easily seen, the pointwise maximal dilatation of f is given by $K(z) = M(|z|)$. Note that the function f can be continuously extended to $z = 0$ by setting $f(0) = 0$ if and only if the integral $\int_0^{\delta_0} dt/tM(t)$ diverges, namely, $\alpha \le 1$. In particular, when $\alpha > 1$, by Theorem A.9 (and Proposition A.22 as well), $\exp(H \circ K(z))$ is not locally integrable around the origin for every dominating factor H of divergence type. On the other hand, we see that $H \circ K$ is exponentially integrable around the origin for $H = cH_{n,\alpha}$, where c is a constant satisfying $c < 2/\eta$. We fix a number ε with $c\eta < \varepsilon < 2$.

From (A.2.5), it follows that $H(M(t)) = cH_{n,\alpha}(\eta \Pi_{n,\alpha}(\log(1/t))) = c\eta(1+o(t))\log(1/t)$ as $t \to +0$. Therefore, there is a number $r_0 > 0$ such that $H(M(t)) \le \varepsilon \log(1/t)$ for all $0 < t < r_0$. We now compute

$$\int_{B(0,r_0)} e^{H(K(z))} d\mathscr{A}(z) = 2\pi \int_0^{r_0} t e^{H(M(t))} dt \le 2\pi \int_0^{r_0} t^{1-\varepsilon} dt < +\infty.$$

In particular, the exponential integrability condition for $H_{n,\alpha} \circ K_\mu$ does not imply the existence of a homeomorphic solution in the case $\alpha > 1$.

Next, we assume that $\alpha < 1$. (As we noted, the case $\alpha = 1$ can be included in the case $\alpha = 0$.) Then the function f can be expressed in the form

$$f(z) = \frac{Cz}{|z|} \exp\left(\frac{-1}{(1-\alpha)\eta} (\log_{n+1}(1/|z|))^{1-\alpha} \right), \quad 0 < |z| < \delta_0.$$

Since η can be chosen arbitrarily whenever $\eta < 2/c$, the sharpness is obtained. □

To state results on modulus bounds of the above dominating factors, we present some auxiliary functions. For constants $C > 1$ and $\beta, \delta > 0$, and a nonnegative integer n, we set

$$\varphi_{n,C}(x) = \exp_n \left(\log_n x + \log C \right)$$

and

$$\psi_{n,\beta,\delta}(x) = \exp_n \left(\left[\left(\log_n x \right)^\beta + \delta \right]^{1/\beta} \right)$$

for $x \ge e_n$. By definition, $\varphi_{n,C} = \psi_{n,1,\log C}$, and, e.g.,

$$\begin{aligned} \varphi_{0,C}(x) &= x + \log C, \\ \varphi_{1,C}(x) &= Cx, \quad \text{and} \\ \varphi_{2,C}(x) &= x^C. \end{aligned}$$

Proposition A.23. *For a positive integer n, a modulus bound for H_n is given by*

$$m(r) = \varepsilon \varphi_{n+1,C}\left(\log 1/r \right)$$

for r small enough, where $\varepsilon > 0$ and $C > 0$ are constants.

Proof. From Lemma A.21, we see that

$$\begin{aligned} &\liminf_{r \to 0} \int_0^{m(r)} \frac{dt}{H_n{}^{-1}(2t - 2\log r)} \\ &= \frac{1}{2} \liminf_{r \to 0} \left[\log_{n+1}\left(2m(r) - 2\log r \right) - \log_{n+1}(-2\log r) \right]. \end{aligned}$$

Noting that $\log r = o(m(r))$ as $r \to 0$, we obtain

$$\begin{aligned}\log_{n+1}\left(2m(r) - 2\log r\right) &= \log_{n+1}(\varphi_{n+1,C}(\log 1/r)) + o(1)\\ &= \log_{n+1}(\log 1/r) + \log C + o(1)\end{aligned}$$

and $\log_{n+1}(m(r)) = \log_{n+1}(\log 1/r) + o(1)$ as $r \to 0$. Hence, $m(r)$ is a modulus bound for H_n. □

The same method yields the following result:

Proposition A.24. *For a positive integer n and* $\alpha \in (0,1)$, *a modulus bound for* $H_{n,\alpha}$ *is given by*

$$m(r) = \varepsilon \psi_{n,1-\alpha,\delta}\left(\log 1/r\right)$$

for r small enough, where $\varepsilon > 0$ *and* $\delta > 0$ *are constants.*

A.3 The Existence and Uniqueness Theorems of Martio–Miklyukov

Existence and uniqueness of solutions of degenerate Beltrami equation were studied in the paper [158] under the condition that the dilatation K has an upper bound in $W^{1,2}$.

A.3.1 Introduction and Main Results

Let $w : D \to \mathbf{R}^2$ be a homeomorphism of a domain $D \subset \mathbf{R}^2$ onto a domain $\mathscr{D} = w(D) \subset \mathbf{R}^2$ and let $\mu : D \to \mathbf{R}^2$ be a measurable function such that $|\mu(z)| < 1$ for a.e. $z \in D$. We say that w is a μ-homeomorphism if $w \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$w_{\bar{z}} = \mu(z) w_z \quad \text{for a.e. } z \in D. \tag{A.3.1}$$

Write

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|};$$

then $K_\mu(z) \in [1, \infty]$ and $K_\mu(z) < \infty$ for a.e. $z \in D$.

By the theorem of Gehring and Lehto [91], a μ-homeomorphism w is a.e. differentiable and (A.3.1) yields $\|w'(z)\|^2 \le K_\mu(z) J(z,w)$ a.e., where $\|w'(z)\|$ is the operator norm $\|w'(z)\| = \sup_{|h|=1} |w'(z)h|$ of the linear map $w'(z)$ and $J(z,w) = \det w'(z)$ is the Jacobian of $w'(z)$.

We recall that a mapping w is q-quasiconformal, $1 \le q < \infty$, if w is a μ-homeomorphism and $K_\mu(z) \le q$ for a.e. $z \in D$. For $q = 1$, the mapping w is conformal. Note that if w is q-quasiconformal, then $w \in W^{1,2}_{\mathrm{loc}}(D)$ and w^{-1} is q-quasiconformal as well.

A measurable function $p : D \to [1,\infty]$ is said to be locally $W^{1,2}$-majorized in D if for every subdomain $U \subset\subset D$ there exists a function $K_U \in W^{1,2}(U)$ such that $p(z) \le K_U(z)$ a.e. in U. Note that if p is locally $W^{1,2}$-majorized, then the set $\{z : \lim_{\xi\to z} p(\xi) = \infty\}$ is of 2-capacity 0 (e.g., see [3, Theorem 6.2.1]).

Theorem A.13. *Suppose that $\mu : D \to \mathbf{R}^2$ is a measurable function such that $|\mu| < 1$ a.e. and that K_μ is locally $W^{1,2}$-majorized in D. Then there exists a μ-homeomorphism $w : D \to \mathbf{R}^2$ onto a domain $\mathscr{D} = w(D)$ with $w \in W^{1,2}_{\mathrm{loc}}(D)$ and $w^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathscr{D})$.*

Moreover, the mapping w is unique up to conformal mappings on the w-plane.

Homeomorphic $W^{1,2}_{\mathrm{loc}}$ solutions of the Beltrami equation (A.3.1) with $K_\mu(z) \le q$ were first systematically studied by Morrey [176]. Bers [34] and Belinskii [30] considered mappings with locally bounded distortion coefficients $K_\mu(z)$. The proof for Theorem A.13 follows from the following result on μ-homeomorphisms of a disk. We let $B = B(0,1)$ denote the unit disk, and the space $W^{1,2}_0(B)$ is the closure of $C^\infty_0(B)$ in $W^{1,2}(B)$.

Theorem A.14. *Let $\mu : B \to \mathbf{R}^2$ be a measurable function such that $|\mu| < 1$ a.e. and that $K_\mu(z) \le K(z)$ a.e. in D, where $K \in W^{1,2}(B)$. Then there exists a μ-homeomorphism w of B onto B such that $w \in W^{1,2}_{\mathrm{loc}}(B)$ and $w^{-1} \in W^{1,2}_{\mathrm{loc}}(B)$. Moreover, w is unique up to conformal transformations on the w-plane. If, in addition,*

$$K_\mu(z) \le K_0 + K(z) \quad \text{a.e. in } D, \tag{A.3.2}$$

where $1 \le K_0 \equiv \mathrm{const} < \infty$ and $K \in W^{1,2}_0(B)$, then $w \in W^{1,2}(B)$ and $w^{-1} \in W^{1,2}(B)$.

The case (A.3.2) of Theorem A.14 was proved by Miklyukov and Suvorov [174]. Because $W^{1,2}$ functions belong to BMO and VMO classes (see [58, Sect. I.2]), then quasiconformal mappings with locally $W^{1,2}$ majorized characteristics are BMO-qc and, respectively, VMO-qc mappings which are studied in [161, 213]. For other results concerning degenerate quasiconformal mappings, see [70, 75, 98, 133, 149, 185, 271], [116, Sect. 10], and others. For the existence, we employ the standard method of cutting the dilatation μ. However, the uniqueness is the essential feature of the paper because without uniqueness the mappings are difficult to handle.

A.3.2 Proof of Theorem A.14

For $n = 1, 2, \ldots$ we set

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \le 1 - 1/n, \\ 0, & \text{if } |\mu(z)| > 1 - 1/n. \end{cases}$$

There exists a mapping $w_n(z)$ from B onto B with dilatation μ_n [152, Sects. 1 and V]. We may assume that $w_n(0) = 0$ for all $n = 1, 2, \ldots$. We call such a sequence w_n a *canonical approximating sequence* for μ.

Because $w_n = (u_n, v_n) \in W^{1,2}(B)$ for every fixed $n = 1, 2, \ldots$ we can find a sequence

$$\{w_{nk}\} = \{(u_{nk}, v_{nk})\}_{k=1}^{\infty} \subset W^{1,2}(B) \cap C^{\infty}(B)$$

such that $w_{nk} \to w_n$ in $W^{1,2}(B)$ or, in other words,

$$\lim_{k\to\infty} \int_B \left(|\nabla(u_{nk} - u_n)|^2 + |\nabla(v_{nk} - v_n)|^2 \right) \mathrm{d}x\,\mathrm{d}y = 0, \tag{A.3.3}$$

$$\lim_{k\to\infty} \int_B \left(|u_{nk} - u_n|^2 + |v_{nk} - v_n|^2 \right) \mathrm{d}x\,\mathrm{d}y = 0. \tag{A.3.4}$$

We may also assume that $u_{nk} \to u_n$ and $v_{nk} \to v_n$ pointwise in B. This is needed later.

Fix n and k. Let ψ be a nonnegative continuous function with a compact support $\operatorname{supp}\psi \subset B$. Then

$$\begin{aligned}
&\left| \int_B \psi J(z, w_n)\,\mathrm{d}x\,\mathrm{d}y - \int_B \psi J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad = \left| \int_B \psi\,(\partial_x u_n \partial_y v_n - \partial_y u_n \partial_x v_n)\,\mathrm{d}x\,\mathrm{d}y \int_B \psi\,(\partial_x u_{nk} \partial_y v_{nk} - \partial_y u_{nk} \partial_x v_{nk})\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \le \int_B \psi\,\big|(\partial_x u_n - \partial_x u_{nk})\,\partial_y v_n - (\partial_y u_n - \partial_y u_{nk})\,\partial_x v_n\big|\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \int_B \psi\,\big|\partial_x u_{nk}\,(\partial_y v_n - \partial_y v_{nk}) - \partial_y u_{nk}\,(\partial_x v_n - \partial_x v_{nk})\big|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus, from the inequality $|ab + cd| \le \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}$, it follows that

$$\begin{aligned}
&\left| \int_B \psi J(z, w_n)\,\mathrm{d}x\,\mathrm{d}y - \int_B \psi J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \le \int_B \psi\,|\nabla(u_n - u_{nk})|\,|\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y + \int_B \psi |\nabla u_{nk}|\,|\nabla(v_n - v_{nk})|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Now, from the Cauchy inequality, we obtain

$$\begin{aligned}
&\left| \int_B \psi J(z, w_n)\,\mathrm{d}x\,\mathrm{d}y - \int_B \psi J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \le \left(\int_B |\nabla(u_n - u_{nk})|^2\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \left(\int_B \psi^2 |\nabla v_n|^2 \mathrm{d}x\,\mathrm{d}y \right)^{1/2} \\
&\qquad + \left(\int_B |\nabla(v_n - v_{nk})|^2\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \left(\int_B \psi^2 |\nabla u_{nk}|^2 \mathrm{d}x\,\mathrm{d}y \right)^{1/2}.
\end{aligned}$$

Note that

$$\int_B \psi^2 |\nabla v_n|^2 \mathrm{d}x\,\mathrm{d}y \le M^2 \int_B |\nabla v_n|^2 \mathrm{d}x\,\mathrm{d}y,$$

where $M = \max_B \psi$, and that

$$\int_B \psi^2 |\nabla u_{nk}|^2 \mathrm{d}x\,\mathrm{d}y \le 2M^2 \int_B |\nabla(u_{nk} - u_n)|^2 \mathrm{d}x\,\mathrm{d}y + 2M^2 \int_B |\nabla u_n|^2 \mathrm{d}x\,\mathrm{d}y.$$

Hence, it follows from (A.3.3) that

$$\int_B \psi J(z, w_n)\,\mathrm{d}x\,\mathrm{d}y = \lim_{k\to\infty} \int_B \psi J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y. \tag{A.3.5}$$

Next, we fix a sequence $\{K_l\}$ of $C^1(B)$ functions with

$$\|K_l - K\|_{W^{1,2}(B)} \to 0 \quad \text{as } l \to \infty. \tag{A.3.6}$$

Let $U \subset\subset B$ be a domain with a rectifiable boundary ∂U. We choose a Lipschitz function $\phi : B \to \mathbf{R}$ with a compact support $\operatorname{supp}\phi \subset B$ such that $\phi = 1$ on U. Let V be a subdomain of B with a smooth boundary ∂V and such that

$$U \subset\subset \operatorname{supp}\phi \subset\subset V \subset\subset B.$$

We set $\psi = \phi^2 K_l$. Then ψ has a compact support in B. The function ψ belongs to the Lipschitz class, so that for $k, l = 1, 2, \ldots$, the Green formula and partial integration yield

$$\begin{aligned}
\int_V \psi J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y &= \int_V \psi\left(\partial_x u_{nk} \partial_y v_{nk} - \partial_y u_{nk} \partial_x v_{nk}\right) \mathrm{d}x\,\mathrm{d}y \\
&= \int_{\partial V} \psi u_{nk}\left(\partial_x v_{nk}\,\mathrm{d}x + \partial_y v_{nk}\,\mathrm{d}y\right) - \int_V \psi u_{nk}\left(\partial_x v_{ky} - \partial_y v_{kx}\right) \mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_V u_{nk}\left(\partial_x \psi \partial_y v_{nk} - \partial_y \psi \partial_x v_{nk}\right) \mathrm{d}x\,\mathrm{d}y \\
&= - \int_B u_{nk}\left(\partial_x \psi \partial_y v_{nk} - \partial_y \psi \partial_x v_{nk}\right) \mathrm{d}x\,\mathrm{d}y \\
&= -2 \int_B \phi u_{nk} K_l \left(\partial_x \phi \partial_y v_{nk} - \partial_y \phi \partial_x v_{nk}\right) \mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_B u_{nk} \phi^2 \left(K_{lx} \partial_y v_{nk} - K_{ly} \partial_x v_{nk}\right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

This leads to the estimate

$$\begin{aligned}
&\int_B \phi^2 K_l J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le 2 \int_B \phi |u_{nk}| |K_l| |\nabla \phi| \, |\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y + \int_B |u_{nk}| \phi^2 \, |\nabla K_l| \, |\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Letting $l \to \infty$ and using (A.3.6) for a fixed $k = 1, 2, \ldots$, we obtain

$$\int_B \phi^2 K J(z, w_{nk})\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq 2\int_B \phi |u_{nk}||K||\nabla\phi|\,|\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y + \int_B |u_{nk}|\phi^2\,|\nabla K|\,|\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y. \quad \text{(A.3.7)}$$

Next, we employ the length and area principle (see, e.g., [247, Sect. 6, Chap. III, Part I]). Let $z', z'' \in U$ be a pair of points with

$$|z'' - z'| < \min\{1, \mathrm{dist}^2(U, \partial B)\}. \quad \text{(A.3.8)}$$

We consider the family of circles $\{S^1(z', r)\}$ with center at z' and radius r,

$$r' = |z'' - z'| < r < r'' = |z'' - z'|^{1/2}.$$

For $f_k = u_{nk}$ or $f_k = v_{nk}$, $k = 1, 2, \ldots$, we have the standard oscillation estimate

$$\int_{r'}^{r''} \mathrm{osc}^2(f_k, S^1(z', r))\frac{\mathrm{d}r}{r} \leq c_1 I(f_k),$$

where c_1 is a constant and $I(f_k) = \int_B |\nabla f_k|^2 \mathrm{d}x\,\mathrm{d}y$. This yields

$$\inf_{r\in(r',r'')} \mathrm{osc}(f_k, S^1(z', r)) \leq c_1^{1/2} I^{1/2}(f_k) \log^{-1/2}|z'' - z'|^{-1/2}.$$

Now, the mappings $w_{nk} = (u_{nk}, v_{nk})$ are homeomorphisms, and thus, each of functions u_{nk}, v_{nk} is monotone in the sense of Lebesgue. The condition (A.3.8) implies that the circles $S^1(z', r)$ lie inside B and hence

$$\mathrm{osc}(f_k, B^2(z', r')) \leq \mathrm{osc}(f_k, S^1(z', r)), \quad r \in (r', r'').$$

From the previous inequalities, it follows that

$$|f_k(z'') - f_k(z')| \leq c_1^{1/2} I^{1/2}(f_k) \log^{-1/2}|z'' - z'|^{-1/2},$$

and hence,

$$|w_{nk}(z'') - w_{nk}(z')| \leq C(w_{nk}) \log^{-1/2}|z'' - z'|^{-1/2} \quad \text{(A.3.9)}$$

for any $z', z'' \in U$ satisfying (A.3.8), and $C(w_{nk}) = 2^{1/2} c_1^{1/2} \max\{I^{1/2}(u_{nk}), I^{1/2}(v_{nk})\}$.

For fixed n, the integrals $I(u_{nk})$and $I(v_{nk})$ are uniformly bounded by (A.3.3). Since the mappings w_{nk} are uniformly bounded as well, it follows from (A.3.9) that the sequence of mappings $\{w_{nk}\}$ is equicontinuous on U. Thus, $w_{nk} \to w_n$ as $k \to \infty$ uniformly on each subdomain U of B compactly contained in B.

For $k \to \infty$ in (A.3.7), the Fatou lemma implies

$$\begin{aligned}\int_B \phi^2 KJ(z,w_n)\,\mathrm{d}x\,\mathrm{d}y &\le \liminf_{k\to\infty}\int_B \phi^2 KJ(z,w_{nk})\,\mathrm{d}x\,\mathrm{d}y\\ &\le 2\lim_{k\to\infty}\int_B \phi|u_{nk}|K|\nabla\phi|\,|\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\lim_{k\to\infty}\int_B |u_{nk}|\phi^2\,|\nabla K|\,|\nabla v_{nk}|\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and now (A.3.5) together with (A.3.3) yields

$$\begin{aligned}&\int_B \phi^2 KJ(z,w_n)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\le 2\int_B \phi|u_n|K|\nabla\phi||\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y+\int_B |u_n|\phi^2\,|\nabla K|\,|\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{A.3.10}$$

Next, we observe that $K_\mu \le K$ implies

$$|\nabla u_n|^2+|\nabla v_n|^2 \le 2K_{\mu_n}J(z,w_n)\le 2KJ(z,w_n).\tag{A.3.11}$$

Combining (A.3.10) and the Cauchy inequality

$$|ab|\le \frac{\varepsilon^2}{2}a^2+\frac{1}{2\varepsilon^2}b^2,\tag{A.3.12}$$

we have

$$\begin{aligned}&\int_B \phi^2\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x\,\mathrm{d}y \le 2\int_B \phi^2 KJ(z,w_n)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\le 4\int_B \phi|u_n|K|\nabla\phi||\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y+2\int_B |u_n|\phi^2\,|\nabla K|\,|\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\le 3\varepsilon^2\int_B \phi^2|\nabla v_n|^2\mathrm{d}x\,\mathrm{d}y+\frac{2}{\varepsilon^2}\int_B |u_n|^2K^2|\nabla\phi|^2\mathrm{d}x\,\mathrm{d}y\\ &\qquad+\frac{1}{\varepsilon^2}\int_B |u_n|^2\phi^2|\nabla K|^2\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We choose $\varepsilon>0$ such that $3\varepsilon^2<1$, e.g. $\varepsilon=1/2$. Then

$$\begin{aligned}&\int_B \phi^2\left(|\nabla u_n|^2+|\nabla v_n|^2\right)\mathrm{d}x\,\mathrm{d}y\\ &\quad\le c_2(\varepsilon)\left(\int_B |u_n|^2K^2|\nabla\phi|^2\mathrm{d}x\,\mathrm{d}y+\int_B |u_n|^2\phi^2|\nabla K|^2\mathrm{d}x\,\mathrm{d}y\right),\end{aligned}$$

where $c_2(\varepsilon)=2\big/(\varepsilon^2(1-3\varepsilon^2))$. Since $\phi\equiv 1$ on U, we have proved

Lemma A.22. *Under the conditions of Theorem A.14, for each subdomain* $U \subset\subset B$ *and each Lipschitz function* $\phi : B \to \mathbf{R}$, $\phi \equiv 1$ *on* $\overline{U}$ *and* $\operatorname{supp}\phi \subset B$, *we have*

$$\int_U \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y \le c_3 \int_{\operatorname{supp}\phi} |u_n|^2 \left(K^2 + |\nabla K|^2\right) \mathrm{d}x\,\mathrm{d}y, \tag{A.3.13}$$

where $c_3 = c_2 \operatorname*{ess\,sup}_{z\in B} \max\{\phi^2(z), |\nabla\phi(z)|^2\}$.

Next, we consider the convergence of w_n as $n \to \infty$. From (A.3.13) and the inequality $|w_n| < 1$, it follows that

$$\int_U \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y \le c_3 \int_B \left(K^2 + |\nabla K|^2\right) \mathrm{d}x\,\mathrm{d}y.$$

Similarly, as in (A.3.9) for $z', z'' \in U$ with $|z'' - z'| < \min\{1, \operatorname{dist}^2(U, \partial B)\}$, we thus have

$$|w_n(z'') - w_n(z')| \le c_4 \log^{-1/2} |z'' - z'|^{-1/2}, \tag{A.3.14}$$

where c_4 is independent of $n = 1, 2, \ldots$.

The estimate (A.3.14) means that the sequence $\{w_n\}_{n=1}^{\infty}$ is equicontinuous on every compact subset of B. By the standard diagonal process, we can now choose a subsequence $\{w_{n'}\}$ which converges locally uniformly to some continuous mapping $w_0 : B \to \mathbf{R}^2$. We shall show that the *limit mapping* w_0 *is a homeomorphism.*

Let $\mu_{n'}(w)$ be the dilatation coefficient of the inverse mapping to $w = w_{n'}(z)$. Since $K_{\mu_{n'}(w)}(w_{n'}(z)) = K_{\mu_{n'}}(z)$, we see that

$$\begin{aligned}
\int_B K_{\mu_{n'}(w)}(w) J(w, w_{n'}^{-1})\,\mathrm{d}u\,\mathrm{d}v &= \int_B K_{\mu_{n'}}(z)\,\mathrm{d}x\,\mathrm{d}y \\
&\le \pi^{1/2} \left(\int_B K^2_{\mu_{n'}}(z)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\le \pi^{1/2} \left(\int_B K^2(z)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} < \infty.
\end{aligned}$$

Now, the quasiconformality of the inverse mapping $(x_{n'}, y_{n'})$ to $w_{n'}$ yields, a.e. in B,

$$x^2_{n'u} + x^2_{n'v} + y^2_{n'u} + y^2_{n'v} \le 2K_{\mu_{n'}}(w) J(w, w_{n'}^{-1}),$$

and hence,

$$\int_B \left(|\nabla x_{n'}|^2 + |\nabla y_{n'}|^2\right) \mathrm{d}u\,\mathrm{d}v \le 2\pi^{1/2} \left(\int_B K^2(z)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} < \infty. \tag{A.3.15}$$

As in (A.3.14), it follows from (A.3.15) that the family $\{w_{n'}^{-1}\}$ of inverse mappings is uniformly continuous on any compact subset of B, and since $w_{n'}(0)=0$ for each n', the limit mapping w_0 of $w_{n'}$ is a homeomorphism.

Sequences $\{w_{n'}\}$ and $\{w_{n'}^{-1}\}$ converge to w and w^{-1}, respectively. Using (A.3.13) and (A.3.15), we conclude that $w \in W^{1,2}_{\mathrm{loc}}(B)$ and $w^{-1} \in W^{1,2}(\mathscr{D})$, $\mathscr{D} = w(B)$.

Next, we employ [98, Proposition 3.1]:

Lemma A.23. *Let a measurable function $\mu : D \to \mathbf{R}^2$ satisfy $|\mu| < 1$ a.e. Suppose that a homeomorphism $w : D \to \mathbf{R}^2$ is a uniform limit of the canonical approximating sequence w_n for μ. If $w \in W^{1,2}_{\mathrm{loc}}(D)$, then w is a μ-homeomorphism a.e. in D.*

The domain $\mathscr{D} = w(B)$ is simply connected, contains the point $w = 0$, and is contained in B. If $\mathscr{D} \neq B$, then we use an auxiliary conformal mapping $f : \mathscr{D} \to B$, $f(0) = 0$, and consider the mapping $\tilde{w} = f \circ w$. Since f is conformal, f is locally bi-Lipschitz and thus $\tilde{w} \in W^{1,2}_{\mathrm{loc}}(B)$. It easily follows that $\tilde{w}^{-1} \in W^{1,2}_{\mathrm{loc}}(B)$.

It remains to consider uniqueness. Assume that $f : B \to B$ and $g : B \to B$ are two μ-homeomorphisms. Because $f \in W^{1,2}_{\mathrm{loc}}(B)$ and $g^{-1} \in W^{1,2}_{\mathrm{loc}}(B)$, then the superposition $f \circ g^{-1}$ belongs to $W^{1,1}_{\mathrm{loc}}(B)$, and a.e. in B has the dilatation $\mu \equiv 0$. Therefore, the mapping $f \circ g^{-1}$ is conformal [116, Corollary 5.3.1].

Finally, we prove that the condition $K_\mu(z) \le K_0 + K(z)$ with $K \in W^{1,2}_0(B)$ implies that $w, w^{-1} \in W^{1,2}(B)$. For this, observe that $K_{\mu_{\tilde{w}}} = K_\mu$, $\tilde{w} = f \circ w$, and hence, we can consider the mapping w instead $\tilde{w}$. Now, this and (A.3.11) yield

$$\begin{aligned}\int_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y &\le 2\int_B K_\mu J(z,w_n)\,\mathrm{d}x\,\mathrm{d}y \\ &\le 2K_0 \int_B J(z,w_n)\,\mathrm{d}x\,\mathrm{d}y + 2\int_B KJ(z,w_n)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The estimate (A.3.10) can now be used with $\phi \equiv 1$ because $K \in W^{1,2}_0(B)$. It follows that

$$\begin{aligned}\int_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y &\le 2K_0\pi + 2\int_B KJ(z,w_n)\,\mathrm{d}x\,\mathrm{d}y \\ &\le 2K_0\pi + 2\int_B |u_n|\,|\nabla K|\,|\nabla v_n|\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and, from $|u_n| < 1$ and from (A.3.12), we obtain

$$\int_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y \le 2K_0\pi + \varepsilon^2 \int_B |\nabla K|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^2}\int_B |\nabla v_n|^2 \mathrm{d}x\,\mathrm{d}y.$$

Choose ε such that $1/\varepsilon^2 < 1$. Now,

$$\left(1 - \frac{1}{\varepsilon^2}\right) \int_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \mathrm{d}x\,\mathrm{d}y \le 2K_0\pi + \varepsilon^2 \int_B |\nabla K|^2\,\mathrm{d}x\,\mathrm{d}y,$$

and thus, the limit mapping w belongs to $W^{1,2}(B)$. Because $K_{\mu_{w^{-1}}}(w(z)) = K_\mu(z)$, it is easy to see that $w^{-1} \in W^{1,2}(B)$ as well. The theorem follows.

A.3.3 Proof of Theorem A.13

Let $\{D_n\}_{n=1}^{\infty}$ be an expanding sequence of subdomains of D which exhaust D and $\overline{D}_n$ is compactly contained in D, i.e.,

$$D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset D_n \cdots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} D_n = D.$$

Set $R_n = \max_{z\in\overline{D}_n} |z|$ and define

$$\mu_n(z) = \begin{cases} \mu(z), & \text{for } z \in D_n, \\ 0, & \text{for } z \in \mathbf{R}^2 \setminus D_n. \end{cases}$$

It is easy to see that Theorem A.14 holds also for the dilatation μ given in any disk $|z| < R$. Fix a point $z_1 \in D_1$, $z_1 \neq 0$. The condition $K_\mu \leq K_U$ a.e. in $U \subset\subset D$ enables us to use Theorem A.14. Then there exists a homeomorphism $w = w_n(z)$, $w_n(0) = 0$, from the disk $|z| < R_n$ onto some disk $|w| < r_n$ having the dilatation μ_n almost everywhere on $|z| < R_n$ and $w \in W^{1,2}_{\text{loc}}(B(R_n))$, $w^{-1} \in W^{1,2}_{\text{loc}}(B(r_n))$ Using an additional similarity of $|w| < r_n$, we choose r_n so that $w_n(z_1) = z_1$.

In each fixed domain D_{n_0}, the mappings $w_n(z)$, $n > n_0$, can be represented in the form $w_n = \phi_n \circ w_{n_0}(z)$, where ϕ_n are conformal mappings in $\tilde{D}_{n_0} = w_{n_0}(D_{n_0})$. Because $\phi_n(0) = 0$ and $\phi_n(z_1) = z_1$, Theorem 4.1 [152] guarantees that there is a subsequence $\{\phi_{n'}\}$ locally uniformly convergent in $\tilde{D}_{n_0}$ to a conformal mapping $\phi : \tilde{D}_{n_0} \to \mathbf{R}^2$. Therefore, $\{w_n\}$ contains a subsequence $\{w_{n'}\}$ of quasiconformal mappings which converges locally uniformly on D_{n_0} to the homeomorphism $w_0 = \phi \circ w_{n_0}$. By Lemma A.23, the mapping w_0 has the dilatation μ a.e. in D_{n_0}.

Using the well-known diagonal process, we find a subsequence $\{w_{n''}\}$ uniformly inside D convergent to some homeomorphism $w : D \to \mathbf{R}^2$ with the dilatation μ. Clearly, the mapping w satisfies both conditions $w \in W^{1,2}_{\text{loc}}(D)$ and $w^{-1} \in W^{1,2}_{\text{loc}}(\mathscr{D})$. This mapping is unique in the indicated class up to conformal transformations.

A.4 The Existence Theorems of Miklyukov–Suvorov

In this section, we give a translation of the known paper [174].

A.4.1 Notations

$L^\alpha(D)$ is the space of all real-valued functions given in a domain D of the complex plane $z = x + iy$ which are integrable with degree $\alpha \geq 1$ and ordinary norm.

$W^1_\alpha(D)$ is the space of functions in D having generalized partial derivations in the Sobolev sense of the class $L^\alpha(D)$ with norm

$$\|u\|_{W^1_\alpha} = \left(\int\limits_{\mathbb{D}} \left(u_x^2 + u_y^2\right)^{\alpha/2} \mathrm{d}\sigma_z \right)^{1/\alpha} = \left(\int\limits_{D} |\nabla u|^\alpha \mathrm{d}\sigma_z \right)^{1/\alpha}.$$

The notation $W^1_\alpha(D)$ is kept also for the space of complex-valued functions $w = f(z) \equiv u(x,y) + \mathrm{i}v(x,y)$ with norm

$$\|f\|_{W^1_\alpha} = \left(\int\limits_{D} \left(|\nabla u|^2 + |\nabla v|^2\right)^{\alpha/2} \mathrm{d}\sigma_z \right)^{1/\alpha}.$$

$\overset{o}{W}{}^1_\alpha(D)$ is the everywhere dense subspace of $W^1_\alpha(D)$ of all infinitely differentiable functions in $W^1_\alpha(D)$ with a compact support in D.

Note that here we consider only plane one-sheeted mappings $w = f(z)$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself.

A.4.2 The main result

Let $p(z)$, $\theta(z)$ be characteristics of quasiconformal mappings f in the Lavrentieff sense: $p(z) \geq 1$ is defined a.e. in $\mathbb{D}$, $\theta(z)$, $0 \leq \theta(z) \leq \pi$, a.e. in the set $\{z : p(z) > 1\}$.[1]

The classical theory of quasiconformal mappings considers the case

$$\operatorname*{ess\,sup}_{\mathbb{D}} p(z) \leq q < \infty.$$

We call such mappings here, as usual, q-quasiconformal.

The basis of the theory of these mappings and the first existence and uniqueness theorem under some functional constraints on the characteristics p and θ originate with Lavrent'ev [144]. Such a theorem in the general case was proved in the papers [31, 43].

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[1]The authors had in mind the characteristics first introduced in [144]:

$$p(z) = \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|},$$

where $\mu(z) = w_{\bar{z}}/w_z$ and $\theta(z) = (\pi + \arg\mu(z))/2$. They appear from infinitesimal considerations: $\mathrm{d}w = \mathrm{d}w_z\,(\mathrm{d}z + \mu(z)\mathrm{d}\bar{z}) = w_z \varepsilon \mathrm{e}^{\mathrm{i}\varphi}\left(1 + \mu(z)\mathrm{e}^{-2\mathrm{i}\varphi}\right)$ where $\mathrm{d}z = \varepsilon \mathrm{e}^{\mathrm{i}\varphi}$, i.e., the infinitesimal circle $|\mathrm{d}z| = \varepsilon$ centered at the point z is transferred under the differential $\mathrm{d}w$ to an infinitesimal ellipse $\mathscr{E} = \mathscr{E}_{z,\varepsilon}$. The geometric sense of $p(z) \geq 1$ is the ratio between the principal axes of $\mathscr{E}$, and $\theta(z)$ is the angle between the great axis of $\mathscr{E}$ and the real axis.

The authors know only two results for the case of the essentially nonbounded characteristics p: the existence and uniqueness theorem of Belinski obtained for locally bounded p (the result was not published) and the theorem of Pesin [185, 186] under the condition of a "sufficiently strong" integrability of p.[2]

Note that, in the latter theorem, the existence is established in a wider (not usual) class: $f \in W_\alpha^1$ for all $\alpha < 2$, $f^{-1} \in W_2^1$ and

$$\int_{\mathbb{D}} \exp[p(z)]^m \mathrm{d}\sigma_z \le M < \infty, \qquad m > 1.$$

The question as to the uniqueness in the given class remains open.

It is known that a q-quasiconformal mapping belongs to the space W_α^1 with some $\alpha(q) > 2$.[3] However, many important properties of such mappings (absolute continuity by Banach, uniqueness etc.) are guaranteed by their (and f^{-1}) property to belong to the space W_2^1. In this connection, the question about finding a class of nonbounded characteristics with the existence theorem in the class W_2^1 is natural.

The following theorem in the given direction is proved in this work:

Theorem A.15. *Let $p(z)$ and $\theta(z)$ be measurable functions giving a distribution of Lavrentieff's characteristics in $\mathbb{D}$, and let*

$$p(z) \le q_0 + p_0(z) \quad a.e., \tag{A.4.1}$$

where q_0 is a constant and $p_0 \in \overset{o}{W}{}^1_\alpha(\mathbb{D})$.

Then there is the unique mapping $f(z)$ of the disk $\mathbb{D}$ onto itself, $f(0) = 0$, $f(1) = 1$, with the given Lavrentieff's characteristics such that f and $f^{-1} \in W_2^1(\mathbb{D})$.

Moreover, we give here a simple proof of an existence theorem (Theorem A.16), generalizing the theorem of Pesin, where the class of admissible characteristics is extended, however, with extending the space of mappings. The uniqueness theorem most likely does not hold. In the conclusion, we discuss the possibility of further generalizations of our results.

A.4.3 The main lemma

Lemma A.24. *If $w = f(z)$ is a q-quasiconformal mapping of $\mathbb{D}$ onto itself with smooth characteristics $p(z)$ and $\theta(z)$, and $p(z)$ satisfies (A.4.1), then*

$$\|f\|^2_{W_2^1} \le 2\,(q_0 + 1)\,\pi + \|p_0\|^2_{W_2^1} \tag{A.4.2}$$

---

[2] Note that the Belinskii theorem mentioned by the authors was published later as Theorem 9 in the book [30].

[3] See [44].

and

$$\|f^{-1}\|^2_{W^1_2} \le (q_0+1)\,\pi + \|p_0\|_{L^1}. \tag{A.4.3}$$

Proof. We obtain

$$\|f\|^2_{W^1_2} \le (q_0+1)\,\pi + \int\limits_{\mathbb{D}} p_0 \,\frac{\partial\,(u,v)}{\partial\,(x,y)}\,\mathrm{d}\sigma_z \tag{A.4.4}$$

from (A.4.1) and the equality

$$|\nabla u|^2 + |\nabla v|^2 = \left(p + \frac{1}{p}\right)\frac{\partial\,(u,v)}{\partial\,(x,y)}.$$

Note that p_0 is integrable as a function in $\overset{o}{W}{}^1_\alpha(\mathbb{D})$. Integrating by parts, we derive

$$\int\limits_{\mathbb{D}} p_0\,\frac{\partial\,(u,v)}{\partial\,(x,y)}\,\mathrm{d}\sigma_z = -\int\limits_{\mathbb{D}} u\,\frac{\partial\,(p_0,v)}{\partial\,(x,y)}\,\mathrm{d}\sigma_z. \tag{A.4.5}$$

Hence,

$$\begin{aligned}\int\limits_{\mathbb{D}} p_0\,\frac{\partial\,(u,v)}{\partial\,(x,y)}\,\mathrm{d}\sigma_z &\le \iint\limits_{\mathbb{D}} |v|\cdot|\nabla p_0|\cdot|\nabla v|\,\mathrm{d}\sigma_z \\ &\le \frac{1}{2}\int\limits_{\mathbb{D}}\left(|\nabla p_0|^2 + |\nabla v|^2\right)\mathrm{d}\sigma_z \le \frac{1}{2}\,\|p_0\|^2_{W^1_2} + \frac{1}{2}\,\|f\|^2_{W^1_2}.\end{aligned}$$

Thus, we obtain (A.4.2) from here and (A.4.4).

If $p^*(w)$ is a characteristic of $f^{-1}(w)$, then

$$\begin{aligned}\|f^{-1}\|^2_{W^1_2} &= \int\limits_{\mathbb{D}}\left(p^* + \frac{1}{p^*}\right)\frac{\partial\,(x,y)}{\partial\,(u,v)}\,\mathrm{d}\sigma_w \le \pi + \int\limits_{\mathbb{D}} p^*\,\frac{\partial\,(x,y)}{\partial\,(u,v)}\,\mathrm{d}\sigma_w \\ &\le \pi + \int\limits_{\mathbb{D}} p^*\mathrm{d}\sigma_z = \pi + q_0\pi + \int\limits_{\mathbb{D}} p_0\,\mathrm{d}\sigma_z,\end{aligned} \tag{A.4.6}$$

because $p^*(w) = p[f^{-1}(w)]$, and thus, (A.4.3) follows from (A.4.6). □

A.4.4 Proof of Theorem A.15

Let $\{(p_n,\theta_n)\}$ $(n=1,2,\ldots)$ be a sequence of smooth characteristics converging a.e. to (p,θ). By the existence theorem for q-quasiconformal mappings, there is a sequence of mappings of $\mathbb{D}$ onto itself, $f_n(0)=0$, $f_n(1)=1$, whose characteristics coincide a.e. with p_n and θ_n, respectively.

The condition (A.4.1) holds for p_n too, and we conclude on the base of Lemma A.24 that the sequences of mappings f_n and f_n^{-1} have equibounded Dirichlet integrals. Hence, the family $\{f_n\}$ is equicontinuous and equiopen inside $\mathbb{D}$ (and even in $\overline{\mathbb{D}}$, see [247], p. 75) and, by Theorem 22 in [247], p. 139, there is a subsequence $\{f_k\},\ k=1,2,\ldots$ that converges uniformly on compact sets in $\mathbb{D}$ to a topological mapping $w=f_0(z)$ of $\mathbb{D}$ onto itself, $f_0(0)=0$, $f_0(1)=1$, such that f_0 and $f_0^{-1}\in W_2^1$.

Repeating word for word the arguments of the Bers theorem (see, e.g., [153], p. 197), it is shown that a.e.

$$p_0(z)=\lim_{k\to\infty} p_k(z)=p(z), \quad \theta_0(z)=\lim_{k\to\infty}\theta_k(z)=\theta(z).$$

To complete the proof, it suffices to establish the uniqueness of a mapping with the given characteristics if f and f^{-1} belong to the Sobolev class W_2^1.

Let $f_j(z)$ $(f_j(0)=0,\ f_j(1)=1,\ j=1,2)$ be two mappings $f_j,\ f_j^{-1}\in W_2^1$ whose characteristics coincide a.e. with $p(z)$ and $\theta(z)$, respectively.

The mapping $z'=T(w)=f_1[f_2^{-1}(w)]$ belongs to W_1^1 (see [153], p. 159) and consequently is absolutely continuous on almost every straight line, which is parallel to axes. Moreover, it has a.e. in $\mathbb{D}$ the circular property. Thus, T is a q-quasiconformal mapping with $q=1$, i.e., T is conformal,[4] and by the normalization $T(w)\equiv w$, and the proof is complete. □

A.4.5 One more theorem

Theorem A.16. *Let measurable functions $p(z)$ and $\theta(z)$ give the distinction of the Lavrentieff characteristics in $\mathbb{D}$ such that*

(a)

$$p(z)\in L^{\beta}(\mathbb{D}), \quad \beta>1;$$

(b)

$$\int_{\delta}^{x}\frac{\mathrm{d}t}{\int_0^{2\pi} p\left(z_0+e^{-t+i\phi}\right)\mathrm{d}\phi}\to\infty, \quad x\to\infty, \tag{A.4.7}$$

at all points $z_0\in\overline{\mathbb{D}}$.

[4] See, e.g., Weyl's lemma in [9].

Then there is a mapping f of the disk $\mathbb{D}$ onto itself, $f(0)=0$, $f(1)=1$, whose characteristics coincide a.e. in $\mathbb{D}$ with p and θ, respectively, and

$$\|f\|_{W^1_{2\beta/(\beta+1)}} \leq \sqrt{2\pi\|p\|_{L^\beta}}, \tag{A.4.8}$$

i.e., $f \in W^1_{2\beta/(\beta+1)}(\mathbb{D})$ and $f^{-1} \in W^1_2(\mathbb{D})$.

In (A.4.7), $p(z)$ is extended by a constant outside of $\mathbb{D}$.

Proof of Theorem A.16. The proof is a combination of known arguments, and hence, we restrict ourselves by an idea.

We construct cutoff $\{p_N(z)\}$ of p and consider N-quasiconformal mappings $\{f_N(z)\}$ of the disk $\mathbb{D}$ onto itself with characteristics p_N and θ. Then there exist constants a, b, c, and d such that, under $|z-z_0| \leq a$,

$$\exp\left[-b\,\frac{\|f_N^{-1}\|^2_{W^1_2}}{|z-z_0|^2}\right] \leq |f_N(z)-f_N(z_0)|$$

$$\leq c\left[\int\limits_{|z-z_0|}^{d} \frac{\mathrm{d}t}{\int_0^{2\pi} p_N\left(z_0+t\mathrm{e}^{\mathrm{i}\phi}\right)t\,\mathrm{d}\phi}\right]^{-1/2}$$

$$\leq c\left[\int\limits_{\ln\frac{1}{a}}^{\ln\frac{1}{|z-z_0|}} \frac{\mathrm{d}t}{\int_0^{2\pi} p_N\left(z_0+\mathrm{e}^{-t+\mathrm{i}\phi}\right)\mathrm{d}\phi}\right]^{-1/2}. \tag{A.4.9}$$

The first inequality is known (see inequality (32) in [247], p. 75), and the latter is a simple version of known inequality (see, e.g., the second inequality in (1) in [265], p. 76).

The condition (a) together with inequalities (A.4.6), (A.4.9) and condition (b) show that the family $\{f_N\}$ is equicontinuous and equiopen at every point $z_0 \in \mathbb{D}$. From here, similarly to the proof of Theorem A.15, we conclude that there is a subsequence $\{f_{N_k}\}$ which converges uniformly on compact sets in $\mathbb{D}$ to a topological mapping f_0 of the disk $\mathbb{D}$ onto itself and such that $\{f_{N_k}^{-1}\}$ also converges uniformly on compact sets in $\mathbb{D}$ to $f^{-1} \in W^1_2$.

Next, we have

$$\int\limits_{\mathbb{D}} \left(|\nabla u_N|^2+|\nabla v_N|^2\right)^{\beta/(\beta+1)} \mathrm{d}\sigma_z \leq 2^{\beta/\beta+1}\int\limits_{\mathbb{D}} \left[p_N\frac{\partial(u_N,v_N)}{\partial(x,y)}\right]^{\beta/\beta+1} \mathrm{d}\sigma_z$$

$$\leq (2\pi)^{\beta/\beta+1}\left(\int\limits_{\mathbb{D}} p^\beta\,\mathrm{d}\sigma_z\right)^{1/(\beta+1)}.$$

We conclude from here that $\|f_N\|_{W^1_{2\beta/\beta+1}}$ are bounded. Applying the uniform convergence of $\{f_{N_k}\}$ to f_0, we obtain that $f_0 \in W^1_{2\beta/\beta+1}$ by a known theorem (see [230], p. 342). $\square$

The convergence of the characteristics is established similarly to the proof of Theorem A.15.

A.4.6 The final conclusions

The condition (b) in Theorem A.16 holds if the characteristic p is integrable in a strong enough way. In such cases, we can determine a degree of equicontinuity of quasiconformal mappings by revising expressions on the right in inequalities (A.4.9) which, under the conditions of Theorem A.16, are retained as $N \to \infty$. Let us illustrate it by the example of the above theorem of J.N. Pesin.

With this aim, let us establish that the condition

$$\int\limits_{\mathbb{D}} \exp\left[p(z)\right]^m \mathrm{d}\sigma_z \le M < \infty, \quad m > 1 \tag{A.4.10}$$

implies the conditions (a) and (b) in Theorem A.16. The property (a) is obvious in this case. Let us prove divergence of the integral (A.4.7) for the case $z_0 = 0$. The consideration of other points is reduced to this case.

Denoting $A(r) = \int_0^{2\pi} rp\left(re^{\mathrm{i}\varphi}\right)\mathrm{d}\varphi$ and applying the Jensen inequality for the convex functions, we obtain

$$\int\limits_{\mathbb{D}} \exp\left[p(z)\right]^m \mathrm{d}\sigma_z \ge 2\pi \int\limits_0^1 r\exp\left[\frac{A(r)}{2\pi r}\right]^m \mathrm{d}r$$

and, consequently,

$$\int\limits_0^1 r\exp\left[\frac{A(r)}{2\pi r}\right]^m \mathrm{d}r \le M_1 < \infty\,. \tag{A.4.11}$$

Setting

$$E_t = \left\{r \in [t,1] : \exp\left[\frac{A(r)}{2\pi r}\right]^m \ge \frac{1}{tr^3}\right\},$$

we have

$$\int\limits_t^1 r\exp\left[\frac{A(r)}{2\pi r}\right]^m \mathrm{d}r \ge \int\limits_{E_t} \frac{\mathrm{d}r}{tr^2} \ge \int\limits_{1-\mathrm{mes}E_t}^1 \frac{\mathrm{d}r}{tr^2}\,.$$

From here, in view of (A.4.11), we obtain that

$$\text{mes } E_t \leq \frac{M_1 t}{1 + M_1 t}\,. \tag{A.4.12}$$

On the set $[t,1]\backslash E_t$, we have that

$$A(r) \leq 2\pi r \left[\log \frac{1}{tr^3}\right]^{1/m}.$$

Hence,

$$\int_t^1 \frac{\mathrm{d}r}{A(r)} \geq \int_{[t,1]\backslash E_t} \frac{\mathrm{d}r}{A(r)} \geq \frac{1}{2\pi} \int_{[t,1]\backslash E_t} \frac{\mathrm{d}r}{r\left[\log \frac{1}{tr^3}\right]^{1/m}}\,.$$

Noting that the integrand is decreasing on $[t,1]$ and taking into account (A.4.12), we have

$$\begin{aligned}\int_t^1 \frac{\mathrm{d}r}{A(r)} &\geq \frac{1}{2\pi} \int_{t+\text{mes}E_t}^1 \frac{\mathrm{d}r}{2\left[\log \frac{1}{tr^3}\right]} \\ &\geq \frac{m}{2\pi(m-1)} \left[\log \frac{1}{t^2} + \log \frac{1+M_1 t}{1+M_1+M_1 t}\right]^{\frac{(m-1)}{m}} - \frac{m}{2\pi(m-1)} \left[\log \frac{1}{t}\right]^{\frac{(m-1)}{m}}.\end{aligned} \tag{A.4.13}$$

Consequently, the integral diverges as $t \to \infty$ and (A.4.13) can be used to concretize the estimate in (A.4.9).

Thus, (A.4.10) guarantees the conditions of Theorem A.16. Since (A.4.10) implies that p is integrable with any degree, we obtain from (A.4.8) that $f_0 \in W^1_{2\beta/(\beta+1)}$ for any $\beta \neq \infty$, i.e., $f \in W^1_\alpha$ for all $\alpha < 2$. Thus, all conclusions of the theorem of J.N. Pesin hold in our case.

A.5 One Example of Iwaniec–Martin

Here, we present a section devoted to one example from [117]; cf. also [116].

Theorem A.17. *Let* $\mathscr{A} : [1,\infty) \to [1,\infty)$ *be a smooth increasing function with* $A(1) = 1$, $\tau A'(\tau) \geq 1$, *and such that*

$$\int_1^\infty \frac{\mathscr{A}(\tau)}{\tau^2}\,\mathrm{d}\tau < \infty\,. \tag{A.5.1}$$

Then there is a Beltrami coefficient μ *compactly supported in the unit disk* $\mathbb{D}$, $|\mu(x)| < 1$, *with the following properties:*

1. *The distortion function*

$$K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$$

   satisfies

$$\int_{\mathbb{D}} \mathrm{e}^{\mathscr{A}(K(z))} \mathrm{d}m(z) < \infty. \tag{A.5.2}$$

2. *The only* $W^{1,1}_{loc}(\mathbb{D})$ *solutions to the Beltrami equation*

$$f_{\bar{z}} = \mu f_z \quad a.e.\ \mathbb{D} \tag{A.5.3}$$

   which are continuous at the origin are the constant functions.

Proof. Given a function $\mathscr{A}$ satisfying (A.5.1), we define a function $K(s)$, $0 < s \leq 1$, via the functional relation

$$K(s)\,\mathrm{e}^{A(K(s))} = \frac{e}{s^2}. \tag{A.5.4}$$

As the map $K \mapsto K\mathrm{e}^{\mathscr{A}(K)}$ is monotone increasing, we see that the solution $K(s)$ is well defined and unique with $K(1) = 1$. In fact, the solution is decreasing and

$$K'(s) = -\frac{K(s)}{s(1+K\mathscr{A}'(K(s)))} \geq -\frac{K(s)}{s} \tag{A.5.5}$$

as $K(s)\mathscr{A}'(K(s)) \geq 1$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(tK(t)) \geq 0,$$

and so upon integration from s to 1, we obtain

$$K(s) \leq \frac{1}{s}. \tag{A.5.6}$$

Then

$$\mathrm{e}^{\mathscr{A}(K(s))} = \frac{e}{s^2 K(s)} \geq \frac{e}{s},$$

$$K(s) \geq \mathscr{A}^{-1}\left(\log\frac{e}{s}\right).$$

From this, we deduce that for all $0 < t \leq 1$

$$\int_0^t \frac{\mathrm{d}s}{sK(s)} \leq \int_0^t \frac{\mathrm{d}s}{s\mathscr{A}^{-1}(\log e/s)} = \int_a^\infty \frac{\mathscr{A}'(\tau)\,\mathrm{d}\tau}{\tau} \leq \int_1^\infty \frac{\mathscr{A}'(\tau)\,\mathrm{d}\tau}{\tau},$$

where $a = \mathscr{A}^{-1}(\log e/t) \geq \mathscr{A}^{-1}(1) = 1$. The next lemma, which follows from integration by parts and a minor estimate, asserts this last integral is finite. □

Lemma A.25. *If $\mathscr{A}$ is an increasing positive function, then*

$$\int_1^\infty \frac{\mathscr{A}(\tau)\,\mathrm{d}\tau}{\tau^2} < \infty \quad \textit{if and only if} \quad \int_1^\infty \frac{\mathscr{A}'(\tau)\,\mathrm{d}\tau}{\tau} < \infty.$$

Thus,

$$\int_0^t \frac{\mathrm{d}s}{sK(s)}$$

is a well-defined bounded increasing function. Next,

$$\int_{\mathbb{D}} \mathrm{e}^{\mathscr{A}(K(|z|))}\mathrm{d}m(z) = 2\pi \int_0^1 \mathrm{e}^{\mathscr{A}(K(s))} s\mathrm{d}s = 2\pi\mathrm{e}\int_0^1 \frac{\mathrm{d}s}{sK(s)} \leq \int_1^\infty \frac{\mathscr{A}(\tau)\,\mathrm{d}\tau}{\tau},$$

so function K is subexponentially integrable. Now, set

$$f(z) = \frac{z}{|z|}\rho(|z|), \tag{A.5.7}$$

where

$$\rho(t) = \exp\left(\int_0^t \frac{\mathrm{d}s}{sK(s)}\right). \tag{A.5.8}$$

Then f is a radial stretching, defined on $\mathbb{D}\setminus\{0\}$. We compute (away from the origin)

$$\begin{aligned}
\dot{\rho}(t) &= \frac{\rho(t)}{tK(t)}, \\
K(z,f) &= \frac{\rho}{|z|\,\dot{\rho}} = K(|z|), \\
|Df(z)| &= \frac{\rho(|z|)}{|z|},
\end{aligned}$$

$$J(z,f) = \frac{\rho^2(|z|)}{K(z)|z|^2},$$

$$\mu_f(z) = -\frac{z}{\bar{z}}\frac{K(z)-1}{K(z)+1}.$$

Notice here that the function f is not continuous at the origin. However, near the origin we have $\rho(|z|) \approx 1$, and therefore, the formula for $|Df|$ gives us the bounds

$$\frac{1}{|z|} \leq |Df(z)| \leq \frac{C}{|z|}, \tag{A.5.9}$$

which yields our bounded solution in weak-$W^{1,2}$.

Notice that $\omega = f(z)$ is a C^∞ diffeomorphism away from the punctured disk onto the annulus $1 < |\omega| < R = \rho(1)$, and we have "cavitation" at the origin.

Finally, we wish to show that there are no nonconstant continuous solutions in $W^{1,1}_{\mathrm{loc}}(\mathbb{D})$. To this end, let $\varepsilon > 0$ and set $r = \rho(\varepsilon)$. Note that $r \to 1$ as $\varepsilon \to 0$. Let g be a $W^{1,1}_{\mathrm{loc}}(\mathbb{D}) \cap C(\mathbb{D})$ solution to (A.5.3) with $\mu = \mu_f$ as above. Set

$$\varphi(z) = g \circ f^{-1} : \{r < |z| < R\} \to g(\{\varepsilon < |z| < 1\}).$$

As $\varphi \in W^{1,1}(\{r < |z| < R\})$, a simple computation shows $\overline{\partial}\varphi = 0$, and the Weyl lemma gives φ holomorphic. This is true for every $\varepsilon > 0$, and so we are provided with an analytic function for which

$$\varphi \circ f(z) = g(z), \quad z \in \mathbb{D} \setminus \{0\}.$$

This equation implies the function φ has the limit $g(0)$ as $|z| \to 1$. Hence, both φ and g are constant.

The existence and uniqueness theorems of Iwaniec–Martin in the Orlich–Sobolev classes can be found in the monographs [26, 116, 117]; see also the corresponding comments in Sects. 4.6 and 8.7 above.

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